

Diffusion for a quantum particle coupled to phonons in $d \geq 3$.

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Abstract: We prove diffusion for a quantum particle coupled to a field of bosons (phonons or photons). The importance of this result lies in the fact that our model is fully Hamiltonian and randomness enters only via the initial (thermal) state of the bosons. This model is closely related to the one considered in [8] but various restrictive assumptions of the latter have been eliminated. In particular, depending on the dispersion relation of the bosons, the present result holds in dimension $d \geq 3$.

1 Introduction

The rigorous derivation of long-time diffusion from first principles of mechanics, be it quantum or classical, remains an inspiring challenge in mathematical physics. To our best knowledge, there are up to this date very few results of this type, see Section 2.5 for a brief review. Recently, in [8], a model was introduced which is quite tractable and for which diffusion was proven in dimension $d \geq 4$. It is a quantum system described by a Hamiltonian of the type

$$H = H_S + H_E + \lambda H_I, \quad \lambda \in \mathbb{R} \quad (1.1)$$

where H_S is the Hamiltonian of a free particle moving on the lattice and it consists of two parts $H_S = H_{\text{kin}} + H_{\text{spin}}$ describing the translational (kinetic) degrees of freedom and a spin degree of freedom, respectively. The Hamiltonian H_E describes a free field of bosons (the environment), and H_I effectuates the coupling between both. The system is started with the environment in a thermal state at inverse temperature β or in a nonequilibrium state (then we have two phonon fields, at different inverse temperatures $\beta_1 \neq \beta_2$). Such models are a paradigm of open quantum systems. The form of the Hamiltonian will be given in Section 2, but let us already list the properties that allow us to handle this model:

- The mass of the particle, or, since we are on a lattice, rather the inverse hopping strength, is chosen large. In (1.1) this is accomplished by choosing H_{kin} in H_S small. This allows a better control of a diagrammatic expansion in real space, since the particle needs a long time to explore a large volume on the lattice.
- Even though the mass is large, the 'mixing rate' that the spin and momentum degrees of freedom of the particle experience due to the interaction with the phonons is not small. This is possible because of the inclusion of the spin-degree of freedom.

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- By choosing the interaction Hamiltonian sufficiently smooth (in the momentum of the phonons) and the dimension d sufficiently large, we ensure that the free space-time correlation functions of the boson field decay at an integrable rate in time. We need them to decay at least as $\mathcal{O}(t^{-(1+\alpha)})$ for large times t with $\alpha > 1/4$. To engineer this in $d \geq 3$, it suffices to choose the dispersion relation of the bosons to be quadratic in the momentum for small momenta, and to cut off the interaction for large momenta.
- By choosing the coupling constant λ small, we have a well controlled Markovian approximation (Lindblad equation) that describes the particle for times of $\mathcal{O}(\lambda^{-2})$. This Markovian approximation serves as a first approximation to the true behavior and we set up an expansion to control the deviations from it.

Under these assumptions we prove that the reduced dynamics of the particle is diffusive. Our proof is based on a renormalization group (RG) method that was developed in [4, 5, 1] to prove diffusion for random walk in a random environment (RWRE). In the present context the random environment is provided by the phonon field. Unlike in the case of RWRE, in the case at hand the particle influences the environment and the reduced dynamics is non-Markovian. However, the Markovian approximation mentioned above provides a starting point for the analysis where a Markovian dynamics is perturbed by a small non-Markovian noise. In units of the weak coupling time scale $\mathcal{O}(\lambda^{-2})$ our model can then be viewed as a (quantum) random walk in a (quantum) random environment. The RG method consists of an iterative scheme to show that on successive larger temporal and spatial scales the random environment becomes smaller and smaller and the dynamics tends to a renormalized Markovian "fixed point". We show that the renormalized noise vanishes in this limit by showing that its (quantum) correlation functions tend to zero. Here we use a formalism developed earlier by us [22] for the confined case, i.e. the proof that the state of a confined quantum system interacting with a similar field as here tends to the equilibrium state.

The difference of the model considered in this paper and the one treated in [8] is that in the latter case an additional condition was imposed on the free boson correlation function that restricts the model to dimensions $d \geq 4$ and to a rather special class of analytic particle-phonon interaction terms. In the context of these models where the particle mass is chosen to scale as $\mathcal{O}(\lambda^{-2})$ it still remains a challenge to treat more generic phonon or photon reservoirs where the temporal correlations decay as $\mathcal{O}(t^{-1})$ (which is the case in $d = 3$ if the dispersion relation is linear for small momenta). To deal with these cases with our method one needs a more careful RG analysis. A much more difficult and interesting problem is to relax the large mass assumption. In this case the control of the corrections to the Markovian approximation seems still beyond current techniques.

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2 The model

We consider a finite, discrete hypercube $\Lambda = \Lambda_L = \mathbb{Z}^d / L\mathbb{Z}^d$ with $L \in \mathbb{N}$. We will take the thermodynamic limit at the end by letting $L \rightarrow \infty$. Since most concepts in the present section depend on the volume Λ , we do not indicate it explicitly.

2.1 Dynamics

2.1.1 Particle

The Hilbert space of the particle is

$$\mathcal{H}_S = \mathcal{S} \otimes l^2(\Lambda) \quad (2.1)$$

where \mathcal{S} is the finite dimensional space of internal degrees of freedom. The Hamiltonian is

$$H_S = H_{\text{spin}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\text{kin}}, \quad H_{\text{kin}} = -\frac{1}{\lambda^{-2}m_p} \Delta \quad (2.2)$$

where Δ is the discrete Laplacian on Λ with periodic boundary conditions, defined by the kernel $\Delta(x, y) = \delta_{|x-y|,1} - 2d\delta_{x,y}$ where $|x-y| = \sum_{i=1}^d |x_i - y_i|$ with $|\cdot|$ the distance on the torus $\mathbb{Z}/L\mathbb{Z}$. The parameter m_p is the mass of

the particle, it is written with λ^{-2} in front of it to remind us that we choose the mass very big by making λ small. For simplicity, we assume that the Hamiltonian H_{spin} is nondegenerate such that we can label its eigenvectors by their eigenvalues, which we denote by $e \in \sigma(H_{\text{spin}})$.

2.1.2 Phonon field

A single phonon is described by the one-particle Hilbert space $l^2(\Lambda)$. It will however be convenient to consider this space in the Fourier-representation, $l^2(\Lambda^*)$ where $\Lambda^* = \frac{2\pi}{L}(\mathbb{Z}^d/L\mathbb{Z}^d)$ is the dual lattice. We consider Λ^* as a subset of \mathbb{T}^d -the d -dimensional torus identified with $[-\pi, \pi]^d$. The Hilbert space of the phonon field is

$$\mathcal{H}_{\text{E}} = \Gamma(l^2(\Lambda^*)) \quad (2.3)$$

where $\Gamma(\mathfrak{h})$ is the symmetric (bosonic) Fock space built on the one-particle Hilbert space \mathfrak{h} .

The Hamiltonian of the phonon field is given by

$$H_{\text{E}} = \sum_{q \in \Lambda^*} \omega(q) a_q^* a_q \quad (2.4)$$

where a_q^*/a_q are the creation/annihilation operators satisfying the canonical commutation relations (CCR)

$$[a_q, a_{q'}^*] = \delta_{q, q'}. \quad (2.5)$$

$\omega(q) \geq 0$ is the frequency of the phonon with momentum q and we take ω a smooth function defined on \mathbb{T}^d . We impose later further conditions on ω .

We will also consider a non-equilibrium setup, in which case we consider two different phonon fields, distinguished by the label $j = 1, 2$. In this case the Hilbert space is $\mathcal{H}_{\text{E}} = \Gamma(l^2(\Lambda^*)) \otimes \Gamma(l^2(\Lambda^*))$ and the Hamiltonian

$$H_{\text{E}} = \sum_{q \in \Lambda^*} (\omega_1(q) a_{q,1}^* a_{q,1} + \omega_2(q) a_{q,2}^* a_{q,2}) \quad (2.6)$$

where $a_{q,1}^*/a_{q,1}$ act on the first tensor in \mathcal{H}_{E} and $a_{q,2}^*/a_{q,2}$ on the second.

We assume the reader is familiar with these basic notions of second quantization and we refer to [9] for more detailed accounts.

2.1.3 Full Hamiltonian

The interaction between particle and phonon field is chosen linear in the creation/annihilation operators. It is of the form

$$H_I = \sum_{q \in \Lambda^*, j=1,2} (W \otimes e^{iqX} \otimes \phi_j(q) a_{q,j}) + \text{h.c.} \quad (2.7)$$

where ‘h.c.’ stands for ‘hermitian conjugate’, W is a Hermitian matrix acting on \mathcal{S} (the space of the internal degree of freedom), X is the multiplication operator with the variable $x \in \mathbb{Z}^d$ on $l^2(\mathbb{Z}^d)$, and $\phi_j(q)$ are “structure factors” that describe the coupling to the phonons. We will take $\phi_j(q)$ the values of smooth functions $\phi_j : \mathbb{T}^d \rightarrow \mathbb{C}$ at $q \in \Lambda^*$. We will impose some further conditions on ϕ_j later.

The fact that the particle couples to both phonon fields via the same W -matrix, is by no means important, and we make it for notational simplicity.

The total Hamiltonian is then

$$H = H_{\text{S}} + H_{\text{E}} + \lambda H_I \quad (2.8)$$

acting on $\mathcal{H} = \mathcal{H}_{\text{S}} \otimes \mathcal{H}_{\text{E}}$. If $\omega_j(q) > 0$ for any $q \in \Lambda^*$, then H_I is an infinitesimal perturbation (in the sense of Banach operator theory) of H_{E} . By a standard application of the Kato-Rellich theorem, H is self-adjoint on the domain of H_{E} (we need not worry about H_{S} since it is bounded). The condition $\omega_j(q) > 0$ is also necessary to have a well-defined finite-volume Gibbs state (see e.g. (2.16)). However, once we take the thermodynamic limit, the only remaining infra-red regularity assumption will be Assumption A (in particular $\omega_j(0)$ may vanish).

2.2 States

States of the combined system and environment are given by density operators $\rho_{\text{SE}} \in \mathcal{B}_1(\mathcal{H}_{\text{S}} \otimes \mathcal{H}_{\text{E}})$ where \mathcal{B}_1 denotes trace class operators. The initial states that we will consider are of the following form

$$\rho_{\text{SE}} = \rho_{\text{S}} \otimes \rho_{\text{E}}^{\text{ref}} \quad (2.9)$$

with

$$\rho_{\text{E}}^{\text{ref}} = \frac{1}{\mathcal{N}} e^{-\beta_1 H_{\text{E}_1}} \otimes e^{-\beta_2 H_{\text{E}_2}}, \quad \mathcal{N} = \text{Tr} (e^{-\beta_1 H_{\text{E}_1}} \otimes e^{-\beta_2 H_{\text{E}_2}}) \quad (2.10)$$

and $\rho_{\text{S}} \in \mathcal{B}_1(\mathcal{H}_{\text{S}})$ is chosen to have support concentrated around the origin in the following sense. Elements of \mathcal{H}_{S} can be represented by functions $\psi(x)$ with $x \in \Lambda \subset \mathbb{Z}^d$ and taking values in the spin space \mathcal{S} . In this basis ρ_{S} is given by a kernel (matrix) $\rho_{\text{S}}(x', x)$ taking values in $\mathcal{B}(\mathcal{S})$. Then we require

$$\rho_{\text{S}}(x', x) = 0, \quad \text{for } |x|, |x'| > R \quad (2.11)$$

for some R .

Since the environment Hamiltonians are quadratic in the creation/annihilation operators, the density matrix $\rho_{\text{E}}^{\text{ref}}$ is a ‘‘Gaussian state’’, sometimes also referred to as a ‘quasifree state’. Moreover, since these density matrices are functions of the environment Hamiltonian, the initial environment state $\rho_{\text{E}}^{\text{ref}}$ is obviously invariant under the free environment evolution:

$$e^{-itH_{\text{E}}} \rho_{\text{E}}^{\text{ref}} e^{itH_{\text{E}}} = \rho_{\text{E}}^{\text{ref}} \quad (2.12)$$

In fact, these two properties, Gaussianity and invariance under the free dynamics, are what we will really use in our analysis, and the specific choice that we made for $\rho_{\text{E}}^{\text{ref}}$ is not important, except for one of our results which will additionally require $\beta_1 = \beta_2$ and where we exploit the fact that the system is in (close to) thermodynamic equilibrium. The dynamics of the density matrix of the entire system is given by

$$\rho_{\text{SE},t} = e^{-itH} (\rho_{\text{S}} \otimes \rho_{\text{E}}^{\text{ref}}) e^{itH} = e^{-itL} (\rho_{\text{S}} \otimes \rho_{\text{E}}^{\text{ref}}) \quad (2.13)$$

where we denote $L := \text{ad}(H)$.

The reduced dynamics of the particle is defined by taking a partial trace Tr_{E} over the environment Hilbert space \mathcal{H}_{E} :

$$\rho_{\text{S},t} = \text{Tr}_{\text{E}} \rho_{\text{SE},t} = \text{Tr}_{\text{E}} (e^{-itL} (\rho_{\text{S}} \otimes \rho_{\text{E}}^{\text{ref}})) := Z_t \rho_{\text{S}} \quad (2.14)$$

All our results will concern the reduced density matrix $\rho_{\text{S},t}$, which means that we only consider particle observables, but it is straightforward to extend the formalism so as to handle observables of the boson field, as well as more general initial states.

2.3 Thermodynamic limit

Since we are ultimately interested in long-time properties, we have to perform the thermodynamic limit to eliminate Poincaré recurrences. We take care of this below, but first we introduce the free boson correlation function which will play an important role in our main assumptions. First, we define the so-called Segal field operators

$$\Phi(x, t) := \sum_{q \in \Lambda^*, j=\{1,2\}} \left(e^{i(qx + \omega_j(q)t)} \phi_j(q) a_{q,j}^* + e^{-i(qx + \omega_j(q)t)} \overline{\phi_j(q)} a_{q,j} \right) \quad (2.15)$$

The correlation function is then defined as

$$\begin{aligned} \zeta(x, t) &:= \text{Tr} [\rho_{\text{E}}^{\text{ref}} \Phi(x, t) \Phi(0, 0)] \\ &= \frac{1}{|\Lambda|} \sum_{q \in \Lambda^*, j=\{1,2\}} |\phi_j(q)|^2 \left(\frac{1}{e^{\beta \omega_j(q)} - 1} e^{i(xq + t\omega_j(q))} + \frac{1}{1 - e^{-\beta \omega_j(q)}} e^{-i(xq + t\omega_j(q))} \right) \end{aligned} \quad (2.16)$$

where the second equality is again a standard exercise in second quantization, and one recognizes the Bose-Einstein distribution $1/(e^{\beta\omega_j} - 1)$. The correlation function $\zeta(x, t)$ will naturally come up in the evaluation of the perturbation series for the dynamics.

To describe the thermodynamic limit, let us indicate the dependence in (2.16) on the volume Λ explicitly by a superscript $\zeta^\Lambda(x, t)$. We then assume that the finite-volume free correlation functions converge

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \zeta^\Lambda(x, t) = \zeta(x, t) \quad (2.17)$$

uniformly on compacts (in $t \in \mathbb{R}$), and that the limiting correlation function $\zeta(x, t)$ is bounded. This will hold for the basic examples of optical and acoustic phonons in Section 2.4 (in the latter case we need to exclude $q = 0$ in the sum in eq. (2.16), cfr. the discussion in Section 2.1.3). Note also that the spaces \mathcal{H}_S for Λ finite are naturally embedded into the \mathcal{H}_S for $\Lambda = \mathbb{Z}^d$ ($L = \infty$). In particular, a density matrix ρ_S^Λ satisfying (2.11), is embedded into $\mathcal{B}_1(\mathcal{H}_S)$ for $L > R$. More generally, we have

Lemma 2.1. *If the initial state is chosen as described above (including in particular the convergence (2.17) and the boundedness of $\zeta(x, t)$) then the limit*

$$\rho_{S,t} := \lim_{\Lambda \nearrow \mathbb{Z}^d} \rho_{S,t}^\Lambda \quad (2.18)$$

exists in $\mathcal{B}_1(\mathcal{H}_S)$ with \mathcal{H}_S defined with $\Lambda = \mathbb{Z}^d$.

2.4 Results

We need an assumption that will guarantee sufficiently fast decay of correlations of the freely evolving phonon field.

Assumption A (Decay(α)). *There is an $\alpha > 0$ such that*

$$\int_{\mathbb{R}^+} dt (1 + |t|)^\alpha \sup_{x \in \mathbb{Z}^d} |\zeta(x, t)| < \infty \quad (2.19)$$

With no loss we suppose $\alpha < 1$.

The next assumption eliminates a drift by requiring that the model be reflection-symmetric and rotation-symmetric (as far as the lattice allows). Let $O_{\mathbb{Z}^d}$ be the subgroup of the orthogonal group $O(d)$ that fixes the lattice, i.e. $O \in O_{\mathbb{Z}^d}$ iff. $u \in \mathbb{Z}^d \Rightarrow Ou \in \mathbb{Z}^d$ and $|Ou|_2 = |u|_2$ with $|u|_2^2 = \sum_{i=1}^d |u_i|^2$.

Assumption B (Symmetries). *The correlation function ζ is $O_{\mathbb{Z}^d}$ -invariant in the sense that*

$$\zeta(Ox, t) = \zeta(x, t), \quad \text{for any } O \in O_{\mathbb{Z}^d}. \quad (2.20)$$

Of course, this assumption is satisfied by choosing the parameters of the Hamiltonians H_E, H_I and the initial state ρ_E^{ref} symmetric. The $O_{\mathbb{Z}^d}$ -invariance of the particle dynamics was already assured by our explicit choice of H_{kin} .

The next assumption assures that the particle is sufficiently well-coupled to the phonon field. To state it, we need some definitions.

First, we assume that the spectrum of H_{spin} is non-degenerate. Then we can label a basis $\psi_e \in \mathcal{S}$ of eigenvectors of H_{spin} by $e \in \sigma(H_{\text{spin}})$ and we define $W_{e,e'} := \langle \psi_e, W\psi_{e'} \rangle_{\mathcal{S}}$ (in general, we write $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ for the scalar product in a Hilbert space \mathcal{H}).

Secondly, let us introduce the set of Bohr frequencies corresponding to the spin Hamiltonian H_{spin} ,

$$\mathcal{E} := \sigma(\text{ad}(H_{\text{spin}})) = \{\varepsilon = e - e'; e, e' \in \sigma(H_{\text{spin}})\} \quad (2.21)$$

We assume the spectrum of $\text{ad}(H_{\text{spin}})$ is also non-degenerate except for the eigenvalue 0 which is $\dim \mathcal{S}$ -fold degenerate. That is, for any $\varepsilon \in \mathcal{E}, \varepsilon \neq 0$, there is a unique pair $(e, e') \subset \sigma(H_{\text{spin}})$ such that $e' - e = \varepsilon$. Next, let

$$\zeta_\omega(x) = \int_{\mathbb{R}} dt e^{-i\omega t} \zeta(x, t) \quad (2.22)$$

be the Fourier transform of the correlation function in time (well-defined by Assumption A). We require that

$$\lim_{|x| \rightarrow \infty} \zeta_\varepsilon(x) = 0 \quad (2.23)$$

for all $0 \neq \varepsilon \in \mathcal{E}$.

From (the infinite volume limit of) (2.16) the distributional Fourier transform in x of ζ_ω is formally

$$\hat{\zeta}_\omega(q) = (-1)^{\text{sgn}(\omega)} 2\pi \sum_{j=1,2} |\phi_j(q)|^2 \frac{1}{e^{\beta_j \omega} - 1} \delta(\omega_j(q) - |\omega|). \quad (2.24)$$

We assume that for all $0 \neq \varepsilon \in \mathcal{E}$ the distribution $\hat{\zeta}_\varepsilon$ defines a finite positive Borel measure on \mathbb{T}^d which we denote by $\hat{\zeta}_\varepsilon(dq)$. It is readily seen that this assumption holds e.g. if we suppose that $\nabla \omega_j$ nowhere vanishes on a neighborhood of the set $\mathcal{M}_{\varepsilon,j} = \{q \in \mathbb{T}^d \mid \omega_j(q) = |\varepsilon|\}$ for $0 \neq \varepsilon \in \mathcal{E}$. For $\omega = 0$, we assume $\hat{\zeta}_\omega = 0$. These conditions hold in the examples below.

We will now define a Markov jump process with state space $\sigma(H_{\text{spin}}) \times \mathbb{T}^d$. The rates $j(e', dk'; e, k)$ of the jump process (jumps are from unprimed to primed variables) are translation invariant in k -space: $j(e', dk'; e, k) = j(e', d(k' - k); e, 0)$ and given by

$$j(e', dk'; e, 0) = |W_{e,e'}|^2 \hat{\zeta}_{e'-e}(dk') \quad (2.25)$$

The main assumption then is that this Markov process is irreducible. We summarize:

Assumption C (Fermi Golden Rule). *Assume*

- The spectra of H_{spin} and $\text{ad}(H_{\text{spin}})$ (apart from the eigenvalue 0) are non-degenerate
- $\zeta_0 \equiv 0$ and for all $0 \neq \varepsilon \in \sigma(\text{ad}(H_{\text{spin}}))$, $\zeta_\varepsilon(x)$ decays at infinity and $\hat{\zeta}_\varepsilon$ defines a finite positive measure.
- The Markov process on $\sigma(H_{\text{spin}}) \times \mathbb{T}^d$ with jump rates (2.25) is irreducible.

Before proceeding to the results, let us give some examples of models for which all our assumptions are satisfied.

Example 1 (optical phonons). Let

$$\omega(q) = (m_{\text{ph}}^2 + \sum_{i=1}^d \sin^2 q_i)^{\frac{1}{2}}, \quad \text{with } m_{\text{ph}} \neq 0. \quad (2.26)$$

This is a typical dispersion relation of an optical phonon branch. The form factor $\phi(q)$ is a smooth function on \mathbb{R}^d which is nonzero only in neighborhood of 0 such that $|\det \text{Hess}(\omega)|$ is bounded below on $\text{Supp} \phi$. Then a standard argument relying on stationary phase estimates yields

$$\sup_x |\zeta(x, t)| \leq C(1 + |t|)^{-d/2} \quad (2.27)$$

Hence Assumption A is satisfied with $\alpha = (d/2) - 1 - \delta$, for any $\delta > 0$. To ensure that Assumption C is satisfied, we choose $\mathcal{S} = \mathbb{C}^2$ with $W = \sigma_x$ and $H_{\text{spin}} = \varepsilon_0 \sigma_z$ where we use the traditional notation $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then a sufficient condition for Assumption C is that $\mathcal{M}_{\varepsilon_0} \cap \text{Supp} \phi$ has positive measure on $\mathcal{M}_{\varepsilon_0}$. With these choices, our equilibrium result ($\beta_1 = \beta_2$) holds for $d \geq 3$, and the non-equilibrium result ($\beta_1 \neq \beta_2$) holds for $d \geq 4$.

Example 2 (acoustical phonons). If one considers acoustical phonons, for example with the above dispersion relation (2.26) with $m_{\text{ph}} = 0$, then, for smooth $\phi(q)$, $\sup_x |\zeta(x, t)| \leq C(1 + |t|)^{-(d-1)/2}$. Hence, the equilibrium result holds for $d \geq 4$ and the nonequilibrium one for $d \geq 5$. From the point of view of our techniques, the important difference with the above example lies not therein that $\inf \omega = 0$, but in the fact that $\omega(q)$ is linear in $|q|$ for small q .

2.4.1 Diffusion

Our most important result concerns diffusion of the particle. To state this somehow concisely, we note that the density matrix $\rho_{S,t}$ determines a probability measure $\mathbb{P}_t(x)$ on \mathbb{Z}^d :

$$\mathbb{P}_t(x) = \text{Tr}_{\mathcal{S}}[\rho_{S,t}(x, x)] = \sum_{e \in \sigma(H_{\text{spin}})} \rho_{S,t}(x, e; x, e)$$

where $\text{Tr}_{\mathcal{S}}$ is the partial trace on $\mathcal{B}(\mathcal{S})$ (below Tr is the full trace on \mathcal{H}_S), and we wrote the matrix $\rho_{S,t}(x', x) \in \mathcal{B}(\mathcal{S})$ in the basis indexed by $\sigma(H_{\text{spin}})$ as $\rho_{S,t}(x', e'; x, e)$. Physically speaking, $\mathbb{P}_t(x)$ is the probability to find the particle at time t on the lattice site $x \in \mathbb{Z}^d$. Equivalently, for an observable $F(X)$

$$\sum_x \mathbb{P}_t(x) F(x) = \text{Tr}[\rho_{S,t} F(X)], \quad F \in l^\infty(\mathbb{Z}^d) \quad (2.28)$$

Hence, the function $\gamma \mapsto \text{Tr}[\rho_{S,t} e^{i\gamma X}]$, figuring in the theorems below, is the characteristic function of the probability measure \mathbb{P}_t .

Theorem 1 (Equilibrium). *Assume that Assumptions B, C and Assumption A with $\alpha > 1/4$, hold, and moreover, that $\beta_1 = \beta_2$. Then, there is a $\lambda_0 > 0$ such that, for $0 < |\lambda| < \lambda_0$, the characteristic function of $\frac{X}{\sqrt{t}}$ converges to a Gaussian: for all $k \in \mathbb{R}^d$*

$$\lim_{t \rightarrow \infty} \text{Tr}[e^{ik \frac{X}{\sqrt{t}}} \rho_{S,t}] = e^{-|k|^2 D^*}, \quad (2.29)$$

for some strictly positive diffusion constant D^* . Moreover, also moments of $\frac{X}{\sqrt{t}}$ converge to moments of the Gaussian. That is, for any multi-index I

$$\lim_{t \rightarrow \infty} \text{Tr}[(\frac{X}{\sqrt{t}})^I \rho_{S,t}] = (-i\partial_k)^I e^{-|k|^2 D^*} \Big|_{k=0}. \quad (2.30)$$

Theorem 2 (Non-Equilibrium). *Assume that Assumptions B, C and Assumption A hold with $\alpha > 1/2$, but possibly $\beta_1 \neq \beta_2$. Then, there is a $\lambda_0 > 0$ such that, for $0 < |\lambda| < \lambda_0$, the conclusions of Theorem 1 hold*

2.4.2 Thermalization and decoherence

We describe some further results that could be of interest. The following result states that some observables tend to an asymptotic value as $t \rightarrow \infty$. Let ρ_{SE}^β be the (finite volume) Gibbs state at inverse temperature β corresponding to the interacting Hamiltonian H , i.e.

$$\rho_{SE}^\beta = \frac{1}{\text{Tr}(e^{-\beta H})} e^{-\beta H} \quad (2.31)$$

Let $A \in \mathcal{B}(\mathcal{H}_S)$ be a translation-invariant observable, that is, its kernel satisfies $A(x', e'; x, e) = A(x' + y, e'; x + y, e)$ for any $y \in \mathbb{Z}^d$ (recall the notation introduced in Section 2.4.1). For simplicity, we assume A to be an infinite-volume observable and whenever necessary, we make it into a finite-volume ($\Lambda = \mathbb{Z}^d / L\mathbb{Z}^d$) observable by restriction, i.e. $A^\Lambda := \mathbb{1}_\Lambda A \mathbb{1}_\Lambda$ with $\mathbb{1}_\Lambda = \mathbb{1}_{\mathcal{S}} \otimes \mathbb{1}_{l^2(\Lambda)}$. Define the equilibrium expectation value of A as

$$\langle A \rangle_\beta = \lim_{\Lambda \nearrow \mathbb{Z}^d} \text{Tr}[\rho_{SE}^\beta (A^\Lambda \otimes \mathbb{1}_E)] \quad (2.32)$$

Note that ρ_{SE}^β depends on Λ , too. The existence of the limit follows easily by the expansions in Section 10.3.

Theorem 3. *Let $A \in \mathcal{B}(\mathcal{H}_S)$ be a translation-invariant particle observable, as described above. Under the conditions of Theorem 1 (including the restriction on $|\lambda|$), we have*

$$\lim_{t \rightarrow \infty} \text{Tr}[\rho_{S,t} A] = \langle A \rangle_\beta, \quad \text{where } \beta := \beta_1 = \beta_2 \quad (2.33)$$

If we drop the condition that $\beta_1 = \beta_2$, then the asymptotic state of the particle is given by a NESS (non-equilibrium steady state):

Theorem 4. *Let A be a translation-invariant particle observable. Under the conditions of Theorem 2, we have*

$$\lim_{t \rightarrow \infty} \text{Tr}[\rho_{S,t} A] = \langle A \rangle_{\text{NESS}}, \quad (2.34)$$

for some linear functional $A \rightarrow \langle A \rangle_{\text{NESS}}$ on translation invariant observables, satisfying $|\langle A \rangle_{\text{NESS}}| \leq \|A\|_{\mathcal{B}(\mathcal{H}_S)}$, $\langle \mathbb{1} \rangle_{\text{NESS}} = 1$ and $\langle A \rangle_{\text{NESS}} \geq 0$ for $A \geq 0$.

The easiest way to give more details on the NESS consists in using the fact that it is a small perturbation (in λ) of the NESS that one obtains in the Markovian approximation to our model, and this will be described below. Let us however quote one important property of both functionals $\langle \cdot \rangle_{\text{NESS}}$ and $\langle \cdot \rangle_{\beta}$, namely decoherence.

Choose the observables A_y with kernel $A_y(x', e', x, e) = \delta_{x'-x, y}$, then there is a $\gamma_0 > 0$

$$\langle A_y \rangle_{\text{NESS}} \leq C e^{-\gamma_0 |y|} \quad (2.35)$$

for some constant C , independent from y . The same statement holds for $\langle A_y \rangle_{\beta}$ as well, but this does not require our analysis since it is a property of the interacting Gibbs state introduced above.

2.4.3 The Markov approximation

As mentioned already above, we can describe our results qualitatively by referring to the Markov approximation to our model. Strictly speaking this Markov approximation retains the quantum nature of the problem, but as far as the long-time properties of the system are concerned, we can describe what is happening with the help of the 'classical' Markov process that was already introduced before Assumption C by specifying the rates $j(\cdot, \cdot)$ on the state space $\mathcal{F} = \sigma(H_{\text{spin}}) \times \mathbb{T}^d$. One should think of the elements in \mathcal{F} as 'good quantum numbers' for the Hamiltonian H_S , with $k \in \mathbb{T}^d$ momentum and $e \in \sigma(H_{\text{spin}})$ spin. Since we assumed it to be irreducible, this Markov process has a unique invariant state given by an absolutely continuous positive measure. We call the associated density $\mu_Q(k, e)$.

If $\beta_1 = \beta_2$, then we simply have

$$\mu_Q(k, e) = (1/2\pi)^d e^{-\beta e} \left(\sum_{e'} e^{-\beta e'} \right)^{-1} \quad (2.36)$$

The fact that this Gibbs state is uniform in k is due to the fact that the kinetic energy was chosen so small that on the time scale λ^{-2} for which the Markov approximation is valid, the kinetic degrees of freedom do not couple directly to the phonons.

For $\beta_1 \neq \beta_2$, we do not have an explicit expression for the invariant density $\mu_Q(k, e)$, but it is connected to the NESS discussed in the previous sections as follows. For a translation invariant A , we define

$$\langle A \rangle_{\text{NESS}, Q} := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} dk \sum_e \mu_Q(k, e) \sum_x e^{ikx} A(0, e, x, e) \quad (2.37)$$

Then we have

$$\langle A \rangle_{\text{NESS}} - \langle A \rangle_{\text{NESS}, Q} = o(|\lambda|^0), \quad \lambda \rightarrow 0 \quad (2.38)$$

In fact, the same statement holds true when $\beta_1 = \beta_2$, but in that case it follows simply by explicitly comparing the (S-part of) the full interacting Gibbs state with $\mu_Q(k, e)$, i.e. no analysis of the dynamics is necessary. Furthermore, we note that by the invariance property $j(e, 'dk'; e, k) = j(e, 'd(k'-k); e, 0)$, we can deduce that $\mu_Q(k, e) = (1/2\pi)^d \mu_Q(e)$ (independent of k).

From the Markov approximation, we can also infer a diffusion constant. Assume that the Markov process on \mathcal{F} describes a particle that jumps in its (k, e) coordinates and, in between jumps, propagates freely in space with velocity

$$v_i = 2m_p^{-1} \sin k_i, \quad k \in \mathbb{T}^d \quad (2.39)$$

(note that this is the derivative of the dispersion relation $m_p^{-1} \sum_{i=1}^d (2 - 2 \cos k_i)$ of $\lambda^{-2} H_{\text{kin}}$). The diffusion constant D_Q of such a particle is given by the velocity-velocity correlation function

$$D_Q \delta_{i,j} = \frac{1}{2} \int_{\mathbb{R}} dt \langle v_i(t) v_j(0) \rangle \quad (2.40)$$

where the expectation $\langle \cdot \rangle$ is computed with respect to the stationary Markov process. Then, the diffusion constant D^* in Theorems 1, 2 has the asymptotics $D^* = \lambda^2 D_Q + o(\lambda^2)$ as $\lambda \rightarrow 0$.

2.5 Related work

2.5.1 Classical mechanics

Diffusion has been established for the two-dimensional finite horizon billiard in [6]. In that setup, a point particle travels in a periodic, planar array of fixed hard-core scatterers. The *finite-horizon condition* refers to the fact that the particle cannot move further than a fixed distance without hitting an obstacle.

In [17], the hard-core scatterers are replaced by a planar lattice of attractive Coulombic potentials, i.e., the potential is $V(x) = -\sum_{j \in \mathbb{Z}^2} \frac{1}{|x-j|}$. In that case, the motion of the particle can be mapped to the free motion on a manifold with strictly negative curvature, and one can again prove diffusion.

Recently, a different approach was taken in [3]: Interpreted freely, the model in [3] consists of a $d = 3$ lattice of confined particles that interact locally with chaotic maps such that the energy of the particles is preserved but their momenta are randomized. Neighboring particles can exchange energy via collisions and one proves diffusive behavior of the energy profile.

2.5.2 Quantum mechanics for extended systems

The earliest result for extended quantum systems that we are aware of, [20], treats a quantum particle interacting with a time-dependent random potential that has no memory (the time-correlation function is $\delta(t)$). Recently, this was generalized in [16] to the case of time-dependent random potentials where the time-dependence is given by a Markov process with a gap (hence, the free time-correlation function of the environment is exponentially decaying). In [21], a quantum particle interacting with independent heat reservoirs at each lattice site was treated. This model also has an exponentially decaying free reservoir time-correlation function and as such, it is very similar to [16]. Notice also that, in spirit, the model with independent heat baths is comparable to the model of [3], but, in practice, it is easier since quantum mechanics is linear.

The most serious shortcoming of these results (except for [8], already discussed in the introduction) is the fact that the assumption of exponential decay of the correlation function in time is unrealistic. In the model of the present paper, the space-time correlation function, $\zeta(x, t)$, is the correlation function of freely-evolving excitations in the reservoir, created by interaction with the particle. Since momentum is conserved locally, these excitations cannot decay exponentially in time t , uniformly in x .

In the Anderson model, the analogue of the correlation function does not decay at all, since the potentials are fixed in time. Indeed, the Anderson model is different from our particle-environment model: diffusion is only expected to occur for small values of the coupling strength, whereas the particle gets trapped (Anderson localization) at large coupling.

Finally, we mention a recent and exciting development: in [11], the existence of a delocalized phase in three dimensions is proven for a supersymmetric model which is interpreted as a toy version of the Anderson model.

2.5.3 Quantum mechanics for confined systems

The theory of confined quantum systems, i.e., multi-level atoms, in contact with quasi-free thermal reservoirs has been intensively studied in the last decade, e.g. by [2, 15, 10, 22]. In this setup, one proves approach to equilibrium for the multi-level atom. Although at first sight, this problem is different from ours (there is no analogue of diffusion), the techniques are quite similar and we were mainly inspired by these results. However, an important difference is that, due to its confinement, the multi-level atom experiences a free reservoir correlation function with better decay

properties than that of our model (if one assume enough infrared regularity) and the 'Markov approximation' to the model has a gap, whereas in the extended system it is diffusive.

2.5.4 Scaling limits

Up to now, most of the rigorous results on diffusion starting from deterministic dynamics are formulated in a *scaling limit*. This means that one does not fix one dynamical system and study its behavior in the long-time limit, but, rather, one compares a family of dynamical systems at different times, as a certain parameter goes to 0. The precise definition of the scaling limit differs from model to model, but, in general, one scales time, space and the coupling strength (and possibly also the initial state) such that the Markovian approximation to the dynamics becomes exact. In the model of the present paper, one can do this by considering the dynamics for times $t = \mathcal{O}(\lambda^{-2})$ and then taking $\lambda \rightarrow 0$. The result is a Markovian approximation which was already referred to in Section 2.4.3. Of course, the point of the present paper is that we go beyond the Markovian approximation and we describe the dynamics for infinite times at fixed λ . There are quite some results on scaling limits in the literature, for example [24, 14, 13, 19, 18, 12], and we do not attempt an overview.

2.6 Strategy of proof

We will now give a road map for the proof of Theorems 1 and 2. It is based on a careful analysis of the long time properties of the reduced particle dynamics given by the operator Z_t defined in eq. (2.14) and extended to infinite volume in Section 5. We view Z_t as a bounded map $Z_t : \mathcal{B}_1(\mathcal{H}_S) \rightarrow \mathcal{B}_1(\mathcal{H}_S)$ i.e.

$$Z_t \in \mathcal{B}(\mathcal{B}_1(\mathcal{H}_S)).$$

The main ingredient will be a proof that in a suitable norm the rescaled large time limit

$$\lim_{t \rightarrow \infty} \mathcal{S}_{\sqrt{t}} Z_t \quad (2.41)$$

exists³. Here $\mathcal{S}_{\sqrt{t}}$ is an operator implementing the scaling of the particle position occurring in Theorem 1 in the space $\mathcal{B}(\mathcal{B}(\mathcal{H}_S))$ explained in detail in Section 3.3. This large time limit is controlled in two steps.

2.6.1 Random walk in random environment

The first step consists of studying the dynamics on the time scale $\mathcal{O}(1/\lambda^2)$. This scale is large enough so that the dissipative effects that we want to exhibit are clearly visible, and small enough such that a simple Duhamel expansion can be controlled. Thus, let us fix $t_0 = \lambda^{-2} \tau_0$ with τ_0 of $\mathcal{O}(1)$. The evolution of the system plus environment up to this time is given by

$$\mathcal{U} := e^{-it_0 L}. \quad (2.42)$$

Denote the reduced time evolution on this scale by

$$T := Z_{t_0}. \quad (2.43)$$

In Section 11.1 we show that for λ small enough,

$$T = e^{t_0(-iL_S + \lambda^2 M)} + \mathcal{O}(|\lambda|^{2\alpha}) \quad (2.44)$$

in an appropriate norm, cfr. Proposition 11.1. Here, $L_S = \text{ad}(H_S)$ and M is the generator of a ‘‘quantum Markov process’’ that is very closely related to the Markov process discussed in Section 2.4.

We will now compare the full evolution (2.42) to the one where the particle dynamics is given by the reduced evolution T and the environment evolves freely:

$$\mathcal{U} = T \otimes e^{-it_0 L_E} + B \quad (2.45)$$

³This statement is of course only true in infinite volume

where $L_E = \text{ad}(H_E)$. This defines the “excitation operator B ” acting on $\mathcal{B}_1(\mathcal{H}_S \otimes \mathcal{H}_E)$. This operator is analyzed in Section 10. There we show that B is small and “weakly correlated” (Propositions 10.1 and 10.2). To explain what we mean by this, consider the full evolution for times t longer than t_0 . Taking $t = Nt_0$ we have

$$e^{-iNt_0L} = \mathcal{U}^N = (T \otimes e^{-it_0L_E} + B)^N. \quad (2.46)$$

We rewrite this

$$\mathcal{U}^N = e^{-iNt_0L_E} (T + B(N))(T + B(N-1)) \dots (T + B(1)) \quad (2.47)$$

where we use the shorthand T for $T \otimes \mathbb{1}$ and $e^{-it_0L_E}$ for $\mathbb{1} \otimes e^{-it_0L_E}$ and define

$$B(\tau) := e^{i\tau t_0L_E} B e^{-i(\tau-1)t_0L_E}. \quad (2.48)$$

Using cyclicity of the trace and the fact $e^{-it_0L_E} \rho_E^{\text{ref}} = \rho_E^{\text{ref}}$, i.e. the invariance of the environment state under the uncoupled dynamics we get

$$Z_{Nt_0} \rho_S = \text{Tr}_E [(T + B(N)) \dots (T + B(2))(T + B(1)) (\rho_S \otimes \rho_E^{\text{ref}})] \quad (2.49)$$

If $B(\tau), \tau = 1, \dots, N$ were set to zero in (2.49), the reduced evolution would be given by $Z_{Nt_0} = T^N$ i.e. by a discrete-time semigroup acting in the system space. This is our ‘quantum random walk’. Using (2.44), it is easy to show that it is diffusive; this is done in Section 11. More precisely, we show that the process generated by M is diffusive and the diffusivity of T^N follows then by simple perturbation theory. The presence of $B(\tau)$ produces time dependence and dependence on the environment variables that are traced over in the end. We wish to think about the latter as time dependent *noise* and the Tr_E as an expectation over the noise. Thus, given $D \in \mathcal{B}(\mathcal{B}_1(\mathcal{H}_S \otimes \mathcal{H}_E))$ we define $\mathbb{E}(D) \in \mathcal{B}(\mathcal{B}_1(\mathcal{H}_S))$ by

$$\mathbb{E}(D) \rho_S := \text{Tr}_E [D(\rho_S \otimes \rho_E^{\text{ref}})]. \quad (2.50)$$

Using this notation we have

$$Z_{Nt_0} = \mathbb{E} \mathcal{U}^N \quad (2.51)$$

together with

$$T = \mathbb{E} \mathcal{U} \quad (2.52)$$

and

$$\mathbb{E} B(\tau) = 0 \quad (2.53)$$

for all τ which follows again from the invariance of ρ_E^{ref} . In Section 3.1 we generalize this expectation to a larger algebra needed to analyze (2.49). Given a set $A = \{\tau_1, \tau_2, \dots, \tau_m\} \subset \{1, 2, \dots, N\}$ with the convention that $\tau_i < \tau_{i+1}$ we define the time-ordered *correlation function*

$$G_A := \mathbb{E} (B(\tau_m) \otimes_S B(\tau_{m-1}) \otimes_S \dots \otimes_S B(\tau_1)) \in [\mathcal{B}(\mathcal{B}(\mathcal{H}_S))]^{\otimes m} \quad (2.54)$$

Here \otimes_S denotes a product that is tensor product in the system space and operator product in the environment space, see Section 3.1. Like in classical probability we introduce in Section 3.2 *connected correlation functions* or cumulants G_A^c in our non-commutative setup. Propositions 10.1 and 10.2 show that these cumulants are small and inherit decay in time from the environment correlation function.

Thus we want to think about $T + B(\tau)$ as transition probability kernels in the system space which are random due to the environment dependence and we want to prove average or “annealed” diffusion for the associated process, a quantum version of random walk in random environment.

2.6.2 Renormalization group

The second step in the control of the large time asymptotics of Z_t is to control the large N asymptotics of (2.51) composed with the scaling. This is achieved using Renormalization Group (RG) method which consists of studying (2.49) in an inductive way. Pick an integer ℓ and define the RG map

$$\mathbf{R}(\mathcal{U}) := \mathbf{S}_\ell(\mathcal{U}^{\ell^2}), \quad (2.55)$$

where the scaling $\mathbf{S}_\ell = \mathbf{S}_\ell \otimes \mathbb{1}$ acts on $\mathcal{B}(\mathcal{B}_1(\mathcal{H}_S))$ only (see Section 3.3). Iterating it n times, and using $(\mathbf{S}_\ell)^n = \mathbf{S}_{\ell^n}$ we get

$$\mathcal{U}_n := \mathbf{R}^n(\mathcal{U}) = \mathbf{S}_{\ell^n}(\mathcal{U}^{\ell^{2^n}}). \quad (2.56)$$

Define now

$$T_n := \mathbb{E}[\mathcal{U}_n]. \quad (2.57)$$

By (2.51) T_n gives the (rescaled) reduced dynamics at time $\ell^{2^n}t_0$:

$$T_n = \mathbf{S}_{\ell^n} Z_{\ell^{2^n}t_0}. \quad (2.58)$$

Thus the existence of the limit (2.41) is closely related to proving (in a sense to be made precise)

$$\lim_{n \rightarrow \infty} T_n = T^* \quad (2.59)$$

This is the content of Proposition 6.3.

T_n is analyzed inductively in n . We define the renormalized noise B_n in analogy with (2.45) by

$$\mathcal{U}_n = T_n \otimes e^{-i\ell^{2^n}t_0 L_E} + B_n. \quad (2.60)$$

B_n is studied through its cumulants $G_{n,A}^c$ defined as for B . The RG map (2.55) leads to a recursion

$$T_{n+1} = \mathcal{F}(T_n, G_{n,\bullet}^c) \quad (2.61)$$

where the map \mathcal{F} is given explicitly in eq. (3.8) and another one for the correlation functions

$$G_{n+1,A}^c = \mathcal{F}_A(T_n, G_{n,\bullet}^c) \quad (2.62)$$

with \mathcal{F}_A given in eq. (3.34). The recursion (2.61) is studied in Section 7 and (2.62) in Sections 8 and 9. For this analysis a choice of suitable norm for $G_{n,A}^c$ is crucial: Norms which respect the structure of the iteration (2.62) are constructed in Section 5.2.

The convergence (2.59) is a consequence of the fact that the map (2.62) contracts the norms of the correlation functions $G_{n,\bullet}^c$ and preserve their temporal decay i.e. the noise is “irrelevant”. For the contraction one needs to study carefully the linearization of the map \mathcal{F}_A (Section 8). A crucial input to this analysis is the existence of two symmetries in the problem: unitarity and reversibility (Sections 4.1 and 4.3) that provide the contractive factors.

3 Renormalization group formalism

In this section, we define the correlation functions G_A and derive the recursion relations for their renormalized versions $G_{n,A}$ as well as for the T_n . We also discuss the symmetries preserved by the RG map. Throughout this section and the next one, all expressions are in finite volume. In particular, all sums that appear are finite. The infinite volume limit and the definitions of the tensor product spaces in that limit will be discussed in Section 5.

3.1 Correlation functions of excitations

We will now define the correlation function (2.54) and explain how the reduced dynamics (2.49) can be expressed in terms of it. We abbreviate

$$\mathcal{R}_S = \mathcal{B}(\mathcal{B}_1(\mathcal{H}_S)), \quad \mathcal{R}_E = \mathcal{B}(\mathcal{B}_1(\mathcal{H}_E)) \quad (3.1)$$

such that $U, B(\tau)$ are elements of $\mathcal{R}_S \otimes \mathcal{R}_E$ and T is an element of \mathcal{R}_S . Define, for $D, D' \in \mathcal{R}_S \otimes \mathcal{R}_E$ the object

$$D \otimes_S D' \in \mathcal{R}_S \otimes \mathcal{R}_S \otimes \mathcal{R}_E$$

as an operator product in E-part and tensor product in S-part. Concretely, let first $D = D_S \otimes D_E$ and $D' = D'_S \otimes D'_E$. Then

$$D \otimes_S D' := D_S \otimes D'_S \otimes D_E D'_E.$$

Extend then by linearity to the whole space $\mathcal{R}_S \otimes \mathcal{R}_E$. Iterating this construction we define for $D_i \in \mathcal{R}_S \otimes \mathcal{R}_E$, $i = 1, \dots, m$

$$D_m \otimes_S \dots \otimes_S D_2 \otimes_S D_1 \in (\mathcal{R}_S)^{\otimes m} \otimes \mathcal{R}_E.$$

We define the ‘expectation’

$$\mathbb{E} : (\mathcal{R}_S)^{\otimes m} \otimes \mathcal{R}_E \rightarrow (\mathcal{R}_S)^{\otimes m}$$

as the partial trace

$$\mathbb{E}(F)(\rho_{S,m} \otimes \dots \otimes \rho_{S,1}) := \text{Tr}_E(F(\rho_{S,m} \otimes \dots \otimes \rho_{S,1} \otimes \rho_E^{\text{ref}})).$$

Hence we have explained the definition (2.54) of G_A .

Note that by (2.53) $G_{\{\tau\}} = 0$, and $G_A = G_{A+\tau}$ since $e^{-it_0 L_E} \rho_E^{\text{ref}} = \rho_E^{\text{ref}}$. It will be convenient to label the \mathcal{R}_S ’s and to drop the subscript S (since we will rarely need \mathcal{R}_E). Let \mathcal{R}_τ , $\tau \in \mathbb{N}$ be copies of \mathcal{R}_S . For $A = \{\tau_1, \dots, \tau_m\}$ with $\tau_i < \tau_{i+1}$, we define \mathcal{R}_A by

$$\mathcal{R}_A := \mathcal{R}_{\tau_m} \otimes \mathcal{R}_{\tau_{m-1}} \otimes \dots \otimes \mathcal{R}_{\tau_1}.$$

One sees that \mathcal{R}_A is naturally isomorphic to $\mathcal{R}^{\otimes m}$ by identifying the right-most tensor factor to \mathcal{R}_1 , the next one to \mathcal{R}_2 , etc. We use this correspondence to consistently view G_A as an element of \mathcal{R}_A in what follows. Consider a collection \mathcal{A} of disjoint subsets of \mathbb{N} , then each of the spaces $\mathcal{R}_{A \in \mathcal{A}}$ is naturally embedded into $\mathcal{R}_{\text{Supp } \mathcal{A}}$ where $\text{Supp } \mathcal{A} = \cup_{A \in \mathcal{A}} A$. Consequently, given a collection of operators $K_A \in \mathcal{R}_A$, we can define

$$\bigotimes_{A \in \mathcal{A}} K_A \in \mathcal{R}_{\text{Supp } \mathcal{A}} \quad (3.2)$$

To express (2.49) in terms of the correlation functions (2.54) we need one more operation, the contraction $\iota : \mathcal{R} \otimes \mathcal{R} \rightarrow \mathcal{R}$. On elementary tensors define

$$\iota(V \otimes V') := VV' \quad (3.3)$$

and then extend linearly. We can extend ι to a map $\mathcal{T} : \mathcal{R}^{\otimes m} \rightarrow \mathcal{R}$ in the obvious way:

$$\mathcal{T}(V_m \otimes V_{m-1} \otimes \dots \otimes V_1) := V_m V_{m-1} \dots V_1 \quad (3.4)$$

i.e. $\mathcal{T} = \iota_{m,m-1} \dots \iota_{3,2} \iota_{2,1}$ where $\iota_{k+1,k} : \mathcal{R}^{\otimes k+1} \rightarrow \mathcal{R}^{\otimes k}$ equals $\iota \otimes \mathbb{1}_{\mathcal{R}^{\otimes k-1}}$. With these definitions eq. (2.49) can be written as

$$Z_{Nt_0} = \mathcal{T}(\mathbb{E}[(T + B(N)) \otimes_S \dots \otimes_S (T + B(2)) \otimes_S (T + B(1))]) \quad (3.5)$$

and expanding the product we end up with desired formula

$$Z_{Nt_0} = \sum_{A \subset \{1, \dots, N\}} \mathcal{T} \left[\bigotimes_{\tau \in A^c} T(\tau) \otimes G_A \right], \quad A^c = \{1, \dots, N\} \setminus A \quad (3.6)$$

where the operators $T(\tau)$ are copies of $T \in \mathcal{R}$ embedded in \mathcal{R}_τ . Note that, for each operator appearing in the product, we have specified the space \mathcal{R}_A or \mathcal{R}_τ in which it acts (indeed, recall that G_A was defined to act in \mathcal{R}_A). In contrast, the order in which we write the factors inside the $\mathcal{T}[\cdot]$ does not have any meaning. For a completely explicit expression of \mathcal{T} (and of $\mathcal{T}_{A'}$, which will be introduced later), we refer to Section 3.5.1.

3.2 Connected correlation functions

The ‘connected correlation functions’ or ‘cumulants’, denoted by G_A^c , are defined to be operators on \mathcal{R}_A satisfying

$$G_{A'} = \sum_{\text{partitions } \mathcal{A} \text{ of } A'} \left(\bigotimes_{A \in \mathcal{A}} G_A^c \right) \quad (3.7)$$

The tensor product in this formula is well-defined since $\mathcal{R}_{A'} = \bigotimes_{A \in \mathcal{A}} \mathcal{R}_A$ whenever \mathcal{A} is a partition of A' . Note that this definition of connected correlation functions reduces to the usual probabilistic definition when all operators that

appear are numbers and the tensor product can be replaced by multiplication. Just as in the probabilistic case, the relations (3.7) for all sets A' fix the operators G_A^c uniquely since the formula (3.7) can be inverted.

Inserting (3.7) into (3.6) we get

$$Z_{Nt_0} = T^N + \sum_{\mathcal{A} \in \mathfrak{B}(I)} \mathcal{T} \left[\left(\bigotimes_{\tau \in I \setminus \text{Supp} \mathcal{A}} T(\tau) \right) \otimes \left(\bigotimes_{A \in \mathcal{A}} G_A^c \right) \right] \quad (3.8)$$

where $I = \{1, 2, \dots, N\}$, $\mathfrak{B}(I)$ is the set of non-empty collections \mathcal{A} of disjoint subsets A of I , and $\text{Supp} \mathcal{A} = \cup_{A \in \mathcal{A}} A$. Formula (3.8) follows from (3.6) by substituting (3.7) since, obviously, any disjoint collection \mathcal{A} is a partition of $\text{Supp} \mathcal{A}$. The term T^N in (3.8) originates from $A = \emptyset$ in (3.6).

3.3 Kernels and rescaling of space

To define the scaling operator introduced in (2.41) we need to write our operators in a suitable basis. First, let us fix a basis for the space $\mathcal{B}_1(\mathcal{H}_S)$. Since we assumed H_{spin} to be non-degenerate, we may label a basis of \mathcal{S} by eigenvalues $e \in \sigma(H_{\text{spin}})$. Hence $\psi \in \mathcal{H}_S = \mathcal{S} \otimes l^2(\Lambda_L)$ may be identified with a function $\psi(x, e)$. Next, in this basis $\rho \in \mathcal{B}_1(\mathcal{H}_S)$ becomes a kernel (matrix)

$$\rho = \rho(x_L, e_L; x_R, e_R),$$

where $x_L, x_R \in \Lambda_L$ and $e_L, e_R \in \sigma(H_{\text{spin}})$ (R/L for “right/left”). The scaling operator will act on a particular combination of x_L and x_R . Therefore we need to introduce new coordinates. Note that the construction below depends on the finite volume Λ (by the parameter L) in a trivial way, and we will not indicate this dependence explicitly.

Let, for a vector $a \in \frac{1}{2}\mathbb{Z}^d$, $[a]$ stand for the vector with components $[a]_i = [a_i]$ (largest integer smaller than a_i). Then we define

$$x := \lfloor \frac{x_L + x_R}{2} \rfloor, \quad v := \lfloor \frac{x_L - x_R}{2} \rfloor \quad (3.9)$$

and

$$\eta := x_L - (x + v) \in \{0, 1\}^d \quad (3.10)$$

such that we have

$$x_L = x + v + \eta \quad x_R = x - v \quad (3.11)$$

The variables e_L, e_R, η will play a minor role in our analysis. Together with v , we gather them in one symbol s which takes values in the set

$$\mathcal{S} := \sigma(H_{\text{spin}}) \times \sigma(H_{\text{spin}}) \times \{0, 1\}^d \times \mathbb{Z}^d / L\mathbb{Z}^d.$$

The new variables (x, s) thus run through the set

$$\mathbb{A}_0 := \mathbb{Z}^d / L\mathbb{Z}^d \times \mathcal{S}, \quad (3.12)$$

and we may identify $\mathcal{B}(\mathcal{H}_S)$ with $\ell^\infty(\mathbb{A}_0)$ i.e. $\rho \in \mathcal{B}_1(\mathcal{H}_S)$ becomes a function

$$\rho = \rho(x, s).$$

Similarly $K \in \mathcal{R}$ is given by a function on $\mathbb{A}_0 \times \mathbb{A}_0$ i.e. a kernel $K(x', s'; x, s)$ which acts on $\mathcal{B}_1(\mathcal{H}_S)$ by

$$(K\rho)(x', s') = \sum_{(x, s) \in \mathbb{A}_0} K(x', s', x, s) \rho(x, s) \quad (3.13)$$

We define a distinguished subset $\mathcal{S}_0 \subset \mathcal{S}$

$$\mathcal{S}_0 := \{s = (e_L, e_R, \eta, v) \in \mathcal{S} \mid e_L = e_R, v = 0, \eta = 0\} \quad (3.14)$$

Note that \mathcal{S}_0 is independent of L , $|\mathcal{S}_0| = \dim \mathcal{S}$, and that

$$\mathrm{Tr} \rho = \sum_{x,s} \rho(x,s) 1_{\mathcal{S}_0}(s) \quad (3.15)$$

The scaling in the renormalization step affects only the variable x . At first, x takes values in $\mathbb{Z}^d/L\mathbb{Z}^d$, but later on it lives on finer lattices. This motivates the definitions

$$\mathbb{X}_n := \ell^{-n}(\mathbb{Z}^d/L\mathbb{Z}^d), \quad \mathbb{A}_n := \mathbb{X}_n \times \mathcal{S} \quad (3.16)$$

To reconstruct the original coordinates x_L, x_R from the (x, s) at scale n , one finds

$$x_L = \ell^n x + v + \eta \quad x_R = \ell^n x - v. \quad (3.17)$$

The scaling map transforms functions on \mathbb{A}_n into functions on \mathbb{A}_{n+1} , i.e. we define

$$S_\ell : l^\infty(\mathbb{A}_n) \rightarrow l^\infty(\mathbb{A}_{n+1}) \quad (3.18)$$

with

$$S_\ell \rho(x, s) = \ell^d \rho(\ell x, s).$$

Let us denote by \mathcal{R}_n the space of kernels K on $\mathbb{A}_n \times \mathbb{A}_n$ and the action

$$\mathbf{S}_\ell : \mathcal{R}_n \rightarrow \mathcal{R}_{n+1}$$

by $\mathbf{S}_\ell K = S_\ell K S_\ell^{-1}$ i.e.

$$(\mathbf{S}_\ell K)(x', s'; x, s) = \ell^d K(\ell x', s'; \ell x, s). \quad (3.19)$$

When summing over a variable $x \in \mathbb{X}_n$, it is natural to take account of the cell volume i.e. consider Riemann sums. We do this by defining

$$\int_{\mathbb{X}_n} dx f(x) := \sum_{x \in \mathbb{X}_n} \ell^{-nd} f(x) \quad (3.20)$$

Also, we use the shorthand $z = (x, s)$ and the notation

$$\int_{\mathbb{A}_n} dz f(z) = \int_{\mathbb{A}_n} dx ds f(x, s) := \sum_{(x,s) \in \mathbb{A}_n} \ell^{-nd} f(x, s). \quad (3.21)$$

These conventions are motivated by the fact that they preserve normalization. Thus we have

$$\int_{\mathbb{X}_{n+1}} dx (S_\ell f)(x) = \int_{\mathbb{X}_n} dx f(x). \quad (3.22)$$

and

$$\mathrm{Tr} \rho = \mathrm{Tr} S_\ell \rho = \int_{\mathbb{A}_n} dx ds \rho(x, s) 1_{\mathcal{S}_0}(s). \quad (3.23)$$

Similarly the sum in (3.13) becomes an integral and the product of two kernels $K_1, K_2 \in \mathcal{R}_n$ becomes

$$(K_2 K_1)(z', z) = \int_{\mathbb{A}_n} dz'' K_2(z', z'') K_1(z'', z) \quad (3.24)$$

We will also need a discrete delta-function

$$\delta(z', z) = \ell^{-nd} \delta_{x', x} \delta_{s', s}, \quad z = (x, s), z' = (x', s') \quad (3.25)$$

which satisfies

$$\int_{\mathbb{A}_n} dz' \delta(z', z) = 1.$$

3.4 RG recursion for T

Let us now study the renormalized reduced dynamics T_n defined in (2.57). T_n belongs to the rescaled space \mathcal{R}_n . We copy the setup of Section 3.1, defining spaces $\mathcal{R}_{n,A}$ as products of $\mathcal{R}_{n,\tau}$, copies of \mathcal{R}_n . We also define, analogously to (2.48),

$$B_n(\tau) := e^{i\tau\ell^{2n}t_0L_E} B_n e^{-i(\tau-1)\ell^{2n}t_0L_E} \quad (3.26)$$

and the time-ordered correlation function

$$G_{n,A} := \mathbb{E}(B_n(\tau_m) \otimes_S B_n(\tau_{m-1}) \otimes_S \cdots \otimes_S B_n(\tau_1)) \in (\mathcal{R}_n)^{\otimes^m}, \quad (3.27)$$

for $A = \{\tau_1, \dots, \tau_m\}$ with $\tau_1 < \tau_2 < \dots < \tau_m$. With these notations, the concepts of Section 3.1 correspond to $n = 0$, i.e. T and $B(\tau)$ introduced there will now be referred to as $T_0, B_0(\tau)$. As in Section 3.1, we embed $G_{n,A}$ into $\mathcal{R}_{n,A}$ and we write $T_n(\tau)$ to denote a copy of T_n embedded in $\mathcal{R}_{n,\tau}$.

The recursion relation for T_n is obtained from the analogue of eq. (2.51):

$$T_{n+1} = \mathcal{S}_\ell \mathcal{T} [\mathbb{E}((T_n + B_n(\ell^2)) \otimes_S \dots (T_n + B_n(2)) \otimes_S (T_n + B_n(1)))] \quad (3.28)$$

which leads as in (3.6) to

$$T_{n+1} = \sum_{A \subset \{1, \dots, \ell^2\}} \mathcal{S}_\ell \mathcal{T} \left[\bigotimes_{\tau \in A^c} T_n(\tau) \otimes G_{n,A} \right] \quad (3.29)$$

To get the analogue of (3.8) define first $I_{\tau'}$, the set of times at scale n associated to a time τ' at scale $n+1$, by

$$I_{\tau'} := \{\ell^2(\tau' - 1) + 1, \ell^2(\tau' - 1) + 2, \dots, \ell^2\tau'\} \quad (3.30)$$

Then

$$T_{n+1} = \mathcal{S}_\ell [T_n^{\ell^2}] + \sum_{\mathcal{A} \in \mathfrak{B}(I_{\tau'})} \mathcal{S}_\ell \mathcal{T} \left[\left(\bigotimes_{\tau \in I_{\tau'} \setminus \text{Supp} \mathcal{A}} T_n(\tau) \right) \otimes \left(\bigotimes_{A \in \mathcal{A}} G_{n,A}^c \right) \right] \quad (3.31)$$

where $\mathfrak{B}(I_{\tau'})$ is the set of non-empty collections \mathcal{A} of disjoint subsets A of $I_{\tau'}$, and $\text{Supp} \mathcal{A} = \cup_{A \in \mathcal{A}} A$. Note that the τ' on the RHS is arbitrary.

3.5 Recursion relations for correlation functions

We derive the recursion for $G_{n+1,A'}^c$, the cumulants at scale $n+1$, in terms of those at scale n . The set of times at scale n that contribute to $G_{n+1,A'}^c$, is

$$I_{A'} := \cup_{\tau' \in A'} I_{\tau'} \quad (3.32)$$

We will need an extension of the contraction operator \mathcal{T} . Given $\tau' \in \mathbb{N}$, let

$$\mathcal{T}_{\tau'} : \mathcal{R}_{n,I_{\tau'}} \rightarrow \mathcal{R}_{n,\tau'}$$

be the contraction as defined in Section 3.1. (followed by the imbedding of \mathcal{R}_n to $\mathcal{R}_{n,\tau'}$). Given a set $A' \subset \mathbb{N}$, we set

$$\mathcal{T}_{A'} := \bigotimes_{\tau' \in A'} \mathcal{T}_{\tau'} : \mathcal{R}_{n,I_{A'}} \rightarrow \mathcal{R}_{n,A'}$$

In words, we contract within the time intervals $I_{\tau'}$. Below we illustrate the action of $\mathcal{T}_{A'}$ by defining it explicitly with kernels.

Again, let $\mathfrak{B}(I_{A'})$ be the set of non-empty collections \mathcal{A} of disjoint subsets A of $I_{A'}$. Then $\mathcal{A} \in \mathfrak{B}(I_{A'})$ induces a graph $\mathcal{G}_{A'}(\mathcal{A})$ on the vertex set A' by the prescription that we connect $\tau'_1, \tau'_2 \in A'$ by an edge whenever there is an $A \in \mathcal{A}$ such that

$$A \cap I_{\tau'_1} \neq \emptyset, \quad \text{and} \quad A \cap I_{\tau'_2} \neq \emptyset, \quad (3.33)$$

This leads to the basic relation between scale n and $n+1$

$$G_{n+1,A'}^c = \sum_{\mathcal{A} \in \mathfrak{B}(I_{A'}) : \mathcal{G}_{A'}(\mathcal{A}) \text{ connected}} \mathcal{S}_\ell \mathcal{T}_{A'} \left[\bigotimes_{A \in \mathcal{A}} G_{n,A}^c \bigotimes_{\tau \notin \text{Supp} \mathcal{A}} T_n(\tau) \right] \quad (3.34)$$

Here, the rescaling \mathcal{S}_ℓ acts on all copies of \mathcal{R}_n , that is, we write \mathcal{S}_ℓ instead of $\mathcal{S}_\ell \otimes \dots \otimes \mathcal{S}_\ell$. We omit the purely combinatorial proof of this relation.

3.5.1 Kernel representation and the map $\mathcal{T}_{A'}$

In Section 3.3, we introduced coordinates $x \in \mathbb{X}_n, s \in \mathcal{S}, z \in \mathbb{A}_n$, such that operators in \mathcal{R}_n can be identified with kernels on $\mathbb{A}_n \times \mathbb{A}_n$. In a similar way, we can identify operators in $\mathcal{R}_{n,A}$ with kernels on $\mathbb{A}_{n,A} \times \mathbb{A}_{n,A}$, where $\mathbb{A}_{n,A}$ is the cartesian product $\times_{\tau \in A} \mathbb{A}_{n,\tau}$ and $\mathbb{A}_{n,\tau}$ are copies of \mathbb{A}_n . Throughout the paper, we denote by z_A, z'_A elements of $\mathbb{A}_{n,A}$, and, hence, for any $K_A \in \mathcal{R}_{n,A}$, we have a kernel $K_A(z'_A, z_A)$.

Consider a partition \mathcal{A} of A' (hence in particular $\mathcal{A} \in \mathfrak{B}(I_{A'})$) and consider a family of kernels $K_A(z'_A, z_A)$ with $A \in \mathcal{A}$. Let us call a set of two consecutive times $\{\tau, \tau + 1\} \in I_{A'}$ a *bulk bond* whenever $\{\tau, \tau + 1\} \subset I_{\tau'}$ for some τ' . Then we have

$$\left(\mathcal{T}_{A'} \left[\bigotimes_{A \in \mathcal{A}} K_A \right] \right) (\tilde{z}'_{A'}, \tilde{z}_{A'}) = \int dz'_{I_{A'}} dz_{I_{A'}} \prod_{\{\tau, \tau+1\} \text{ bulk bonds}} \delta(z_{\tau+1}, z'_\tau) \prod_{A \in \mathcal{A}} K_A(z'_A, z_A) \quad (3.35)$$

$$\prod_{\tau' \in A'} \delta(\tilde{z}'_{\tau'}, z'_{\max I_{\tau'}}) \delta(\tilde{z}_{\tau'}, z_{\min I_{\tau'}}) \quad (3.36)$$

where the discrete δ -function was defined in (3.25).

4 Unitarity and Reversibility

The correlation functions $G_{n,A}^c$ satisfy identities that are crucial for the proof of our results. The first of these ‘‘Ward identities’’ (called the ‘unitarity’ identity below) is due to invariance of the trace $\text{Tr}(\cdot)$ on $\mathcal{B}_1(\mathcal{H})$ under unitary maps and time-independence on the reference state of the environment w.r.t. the uncoupled dynamics. The second one (the ‘Gibbsian’ identity) is due to the invariance of the equilibrium Gibbs state under the dynamics. To discuss the latter we need to elaborate on the relationship between free and interacting Gibbs states, see Section 4.2.1. The Ward identities will produce some additional smallness in our expansion. The ‘Unitarity’ identity eliminates certain terms exactly, the ‘Gibbsian’ identity does this only approximatively, as long as the temperature is not infinite.

4.1 Ward identity from unitarity

We start by examining the consequences of the conservation of probability, i.e. unitarity of e^{-itH} . Let $\rho_{\text{SE}} \in \mathcal{B}_1(\mathcal{H})$ and abbreviate $\rho = S_{\ell^n-1} \rho_{\text{SE}}$. Then

$$\text{Tr } B_n(\tau) \rho = 0. \quad (4.1)$$

Indeed, from (2.60), (2.55) and (3.26) we infer

$$B_n(\tau) \rho = S_{\ell^n}(U_n \rho U_n^{-1}) - (T_n \otimes \mathbb{1}_E) \rho$$

where $U_n \in \mathcal{B}(\mathcal{H})$ is unitary. Thus, by using (3.23),

$$\begin{aligned} \text{Tr } B_n(\tau) \rho &= \text{Tr}(U_n \rho U_n^{-1}) - \text{Tr}((T_n \otimes \mathbb{1}_E) \rho) \\ &= \text{Tr } \rho - \text{Tr } \rho = 0 \end{aligned} \quad (4.2)$$

By picking appropriate ρ_{SE} this identity leads to the following one for kernels:

$$\int_{\mathbb{A}_n} dz'_\tau 1_{\mathcal{S}_0}(s'_\tau) G_{n,A}^c(z'_A, z_A) = 0, \quad \tau = \max A, z'_\tau = (x'_\tau, s'_\tau). \quad (4.3)$$

We rewrite this relation in the form in which it will be used. Let $P \in \mathcal{B}(l^\infty(\mathcal{S}))$ be a one-dimensional projector with kernel

$$P(s', s) = \mu(s') 1_{\mathcal{S}_0}(s), \quad (4.4)$$

where $\mu(\cdot)$ is some function that satisfies (to ensure that P is indeed a projector) the normalization condition:

$$\sum_{s' \in \mathcal{S}_0} \mu(s') = 1. \quad (4.5)$$

In the rest of the paper, rank-one operators like P will be written as $P = |\mu\rangle\langle 1_{S_0}|$. Then the Ward identity can also be expressed simply as

$$\int dx'_\tau (P \otimes 1 \dots \otimes 1) G_{n,A}^c(x'_A, x_A) = 0, \quad \tau = \max A \quad (4.6)$$

where P acts on (the s -coordinates of) $\mathcal{R}_{\tau=\max A}$.

4.2 Equilibrium setup: the case $\beta_1 = \beta_2$

In this section, we assume that the reference state of the two environments is in equilibrium at temperature $\beta = \beta_1 = \beta_2$.

4.2.1 Equilibrium states

We already introduced the state ρ_E^{ref} on $\mathcal{B}(\mathcal{H}_E)$. We now introduce a corresponding finite-volume reference state on $\mathcal{B}(\mathcal{H}_S)$. It is given by

$$\rho_S^{\text{ref}} = (\text{Tr}[e^{-\beta H_S}])^{-1} e^{-\beta H_S} \quad (4.7)$$

and we define

$$\rho_{SE}^{\text{ref}} = \rho_S^{\text{ref}} \otimes \rho_E^{\text{ref}} \quad (4.8)$$

We will also need the Gibbs state of the coupled system. We introduce

$$\rho_{SE}^\beta = (\text{Tr}[e^{-\beta H}])^{-1} e^{-\beta H}, \quad \rho_S^\beta := \text{Tr}_E \rho_{SE}^\beta \quad (4.9)$$

Hence we adopt the convention of using the superscript “ref” for states that are Gibbs with respect to the uncoupled Hamiltonian, and the superscript β for interacting Gibbs states and restrictions of those. In particular, note that ρ_S^β is not a Gibbs state but merely a restriction of a Gibbs state.

To ease the discussion of the thermodynamic limit, we also introduce

$$\nu^{\text{ref}} := |\Lambda| \rho_S^{\text{ref}}, \quad \nu^\beta := |\Lambda| \rho_S^\beta \quad (4.10)$$

As a consequence of translation invariance, the kernels of $\nu^\beta, \nu^{\text{ref}}$ are constant in the variable $x \in \mathbb{X}_0$, hence they reduce to functions of $s \in \mathcal{S}$;

$$\nu^\beta(x, s) = \mu^\beta(s), \quad \nu^{\text{ref}}(x, s) = \mu^{\text{ref}}(s) \quad (4.11)$$

Moreover, the chosen normalization ensures that

$$\sum_{s \in \mathcal{S}_0} \mu^\beta(s) = \sum_{s \in \mathcal{S}_0} \mu^{\text{ref}}(s) = 1 \quad (4.12)$$

Next, we introduce the Radon-Nikodym derivative as an unbounded operator on \mathcal{H}

$$D_{\text{rd}} := e^{\frac{1}{2}\Delta F(\beta)} e^{-\frac{\beta}{2}H} e^{\frac{\beta}{2}(H_S + H_E)}, \quad (4.13)$$

where $\Delta F(\beta)$ is a number defined by

$$e^{\Delta F(\beta)} = \frac{\text{Tr}[e^{-\beta(H_S + H_E)}]}{\text{Tr}[e^{-\beta H}]} \quad (4.14)$$

One can check that the ‘free energy difference’ $\Delta F(\beta)$ has a limit as $\Lambda \nearrow \mathbb{Z}^d$ (it is not proportional to $|\Lambda|$, because the number of particles does not grow with the volume). D_{rd} is an unbounded operator, even in finite volume, but it is obviously well-defined by the functional calculus. The reason why we call D_{rd} a ‘Radon-Nikodym derivative’ is that, formally,

$$\rho_{SE}^\beta = D_{\text{rd}} \rho_{SE}^{\text{ref}} D_{\text{rd}}^* \quad (4.15)$$

4.2.2 Correlation functions involving $\tau = 0$

Let us define the operator $\mathcal{U}_0(0)$ by

$$\mathcal{U}_0(0)\psi := D_{\text{rd}}\psi D_{\text{rd}}^* \quad (4.16)$$

such that formally

$$\rho_{\text{SE}}^\beta = \mathcal{U}_0(0)\rho_{\text{SE}}^{\text{ref}}. \quad (4.17)$$

It is not hard to see that $\mathcal{U}_0(0)$ is densely defined on $\mathcal{B}_1(\mathcal{H})$. This is however not our point. Rather we want to treat the operator $\mathcal{U}_0(0)$ en par with the operator \mathcal{U}_0 introduced previously, hence we write

$$\mathcal{U}_0(0) = T_0(0) \otimes \mathbb{1}_{\text{E}} + B_0(0) \quad (4.18)$$

where

$$T_0(0) := \mathbb{E} \mathcal{U}_0(0) \quad (4.19)$$

We can now extend formally the definition (2.54) such that the set A can include $\tau = 0$. We define an additional copy, $\mathcal{R}_{n,0}$, of \mathcal{R}_n such that again $G_{n,A}$ is embedded into $\mathcal{R}_{n,A}$. When defining the renormalization transformation of $\mathcal{U}_0(0)$, we omit the iteration and keep the rescaling only, such that we have

$$\mathcal{U}_n(0) := \mathcal{S}_{\ell^n} \mathcal{U}_0(0), \quad B_n(0) := \mathcal{S}_{\ell^n} B_0(0), \quad T_n(0) := \mathcal{S}_{\ell^n} T_0(0) \quad (4.20)$$

With these definitions, we can also formally define the correlation functions $G_{n,A} \in \mathcal{R}_{n,A}$ at higher scales $n \geq 1$.

Lemma 4.1. *The operators $T_n(0)$ and $G_{n,A}, G_{n,A}^c$ with $A \ni 0$, are bounded. The relation (3.34) remains valid for $A \ni 0$ if one defines $I_0 := \{0\}$.*

The boundedness will be easily proven with help of expansions in Section 10.3. The validity of the recursion relation is formally obvious. For later use, we also define the bounded map $\check{Z}_t : \mathcal{B}_1(\mathcal{H}_{\text{S}}) \rightarrow \mathcal{B}_1(\mathcal{H}_{\text{S}})$

$$\check{Z}_t \rho_{\text{S}} := \text{Tr}_{\text{E}}[e^{-itL}(D_{\text{rd}}\rho_{\text{S}} \otimes \rho_{\text{E}}^{\text{ref}} D_{\text{rd}}^*)] = \text{Tr}_{\text{E}}[e^{-itL}\mathcal{U}_0(0)(\rho_{\text{S}} \otimes \rho_{\text{E}}^{\text{ref}})] \quad (4.21)$$

4.3 Ward identity from equilibrium

To state this Ward identity, it is convenient to introduce an additional piece of notation. Given $K \in \mathcal{R} \otimes \mathcal{R}$ and $\psi \in \mathcal{B}_1(\mathcal{H}_{\text{S}})$, let $K\psi \in \mathcal{R} \otimes \mathcal{B}_1(\mathcal{H}_{\text{S}})$ be given by

$$(V_2 \otimes V_1)\psi := V_2 \otimes (V_1\psi) \quad (4.22)$$

on elementary tensors $K = V_2 \otimes V_1$ and then extending by linearity. This notation is only used in the lemma below and its proof.

Let us also define the scaled versions of the states (4.10)

$$\nu_n^{\text{ref}} := \mathcal{S}_{\ell^n} \nu^{\text{ref}}, \quad \nu_n^\beta := \mathcal{S}_{\ell^n} \nu^\beta \quad (4.23)$$

Then we have the Ward identity:

Lemma 4.2.

$$\mathbb{E}(B_n(\tau) \otimes_{\text{S}} B_n(1))\nu_n^\beta = -\mathbb{E}(B_n(\tau) \otimes_{\text{S}} B_n(1)B_n(0))\nu_n^{\text{ref}} - \mathbb{E}(B_n(\tau) \otimes_{\text{S}} (T_n \otimes \mathbb{1}_{\text{E}})B_n(0))\nu_n^{\text{ref}} \quad (4.24)$$

$$+ \mathbb{E}(B_n(\tau - 1) \otimes_{\text{S}} B_n(0))\nu_n^{\text{ref}} \quad (4.25)$$

Proof. Up to trivial rescaling, the proof is the same for all n , we will therefore for simplicity set $n = 0$ and omit it everywhere. We also abuse notation by writing $T_n, T_n(0)$ for $T_n \otimes \mathbb{1}_{\text{E}}, T_n(0) \otimes \mathbb{1}_{\text{E}}$. We introduce the projector \mathcal{P} that acts on $\mathcal{B}_1(\mathcal{H})$ by

$$\mathcal{P}\psi = (\text{Tr}_{\text{E}} \psi) \otimes \rho_{\text{E}}^{\text{ref}}, \quad \psi \in \mathcal{B}_1(\mathcal{H}) \quad (4.26)$$

We start from

$$B(1)\mathcal{P}\rho_{\text{SE}}^\beta = -B(1)(1-\mathcal{P})\rho_{\text{SE}}^\beta - T\rho_{\text{SE}}^\beta + e^{it_0 L_E}\rho_{\text{SE}}^\beta \quad (4.27)$$

where we used $B(1) = e^{it_0 L_E}\mathcal{U} - T$ and $\rho_{\text{SE}}^\beta = \mathcal{U}\rho_{\text{SE}}^\beta$. Next, we substitute

$$\rho_{\text{SE}}^\beta = \mathcal{U}(0)\rho_{\text{SE}}^{\text{ref}} = \mathcal{U}(0)(\rho_{\text{S}}^{\text{ref}} \otimes \rho_{\text{E}}^{\text{ref}}) = (T(0) + B(0))(\rho_{\text{S}}^{\text{ref}} \otimes \rho_{\text{E}}^{\text{ref}})$$

and use $\mathcal{P}\rho_{\text{SE}}^\beta = \rho_{\text{S}}^\beta \otimes \rho_{\text{E}}^{\text{ref}}$ to obtain

$$\begin{aligned} B(1)(\rho_{\text{S}}^\beta \otimes \rho_{\text{E}}^{\text{ref}}) &= (-B(1)(1-\mathcal{P})B(0) - TB(0) + e^{it_0 L_E}B(0))(\rho_{\text{S}}^{\text{ref}} \otimes \rho_{\text{E}}^{\text{ref}}) \\ &+ (e^{it_0 L_E}T(0) - TT(0))(\rho_{\text{S}}^{\text{ref}} \otimes \rho_{\text{E}}^{\text{ref}}) \end{aligned} \quad (4.28)$$

This relation has the form $\sum_j D_j(\psi_{\text{S},j} \otimes \rho_{\text{E}}^{\text{ref}}) = 0$, for some $\psi_{\text{S},j} \in \mathcal{B}_1(\mathcal{H}_{\text{S}})$ and D_j operators on $\mathcal{B}_1(\mathcal{H})$, hence it implies that

$$\sum_j \mathbb{E}(D' \otimes_{\text{S}} D_j)\psi_{\text{S},j} = 0, \quad \text{for any } D' \quad (4.29)$$

where we used the convention (4.22). We choose $D' = B(\tau)$ and we spell out (4.29), obtaining

$$\begin{aligned} \mathbb{E}(B(\tau) \otimes_{\text{S}} B(1))\rho_{\text{S}}^\beta &= -\mathbb{E}(B(\tau) \otimes_{\text{S}} B(1)B(0))\rho_{\text{S}}^{\text{ref}} \\ &- \mathbb{E}(B(\tau) \otimes_{\text{S}} TB(0))\rho_{\text{S}}^{\text{ref}} + \mathbb{E}(B(\tau-1) \otimes_{\text{S}} B(0))\rho_{\text{S}}^{\text{ref}} \end{aligned} \quad (4.30)$$

where the two first terms in (4.28) containing no $B(\cdot)$ -operator have disappeared because $\mathbb{E}(B(\{\tau\})) = 0$, and we used $\mathbb{E}(B(\tau)e^{it_0 L_E} \otimes_{\text{S}} \dots) = \mathbb{E}(B(\tau-1) \otimes_{\text{S}} \dots)$. The lemma follows upon multiplying with $|\Lambda|$ and reinstating the subscript n . \square

4.3.1 Ward identity in terms of correlation functions $G_{A,n}^c$

As it stands, Lemma 4.2 is not written in terms of the correlation functions G_A^c . We can however easily rewrite it in that way. Since the Ward identity compares operators in different copies of \mathcal{H}_n , it is unnatural to index the z, z' -coordinates by $\tau \in A$ and hence we use arbitrary indices, with the convention that the z, z' -coordinates corresponding to earlier times stand to the left of the later ones. We again drop the index n .

The Ward identity reads

$$\int dz_a G_{\{1,\tau\}}^c(z'_a, z'_b; z_a, z_b) \nu^\beta(z_a) = \int dz_a L_\tau(z'_a, z'_b; z_a, z_b) \nu^{\text{ref}}(z_a) \quad (4.31)$$

with

$$\begin{aligned} L_\tau(z'_a, z'_b; z_a, z_b) &:= - \int dz G_{\{0,1,\tau\}}^c(z, z'_a, z'_b; z_a, z, z_b) \\ &- \int dz T(z'_a, z) G_{\{0,\tau\}}^c(z, z'_b; z_a, z_b) + G_{\{0,\tau-1\}}^c(z'_a, z'_b; z_a, z_b) \end{aligned} \quad (4.32)$$

The verification of this relation from Lemma 4.2 is by inspection, using the fact that $G_A = G_A^c$ whenever $|A| \leq 3$, which in turn follows from the vanishing of G_A whenever $|A| = 1$.

Remark. The reader should think of eq. (4.31) as analogous to the unitarity Ward identity, eq. (4.3). Namely, the RHS turns out to run down to zero fast under the RG (i.e. as $n \rightarrow \infty$) and this lets us get extra contraction for the RG flow of the two-point function.

5 The infinite volume setup

Up to this point, our whole treatment was restricted to finite volume $\mathbb{Z}^d/L\mathbb{Z}^d$. This allowed us to neglect all sorts of analytical questions since all the operators pertaining to the S-part were actually finite matrices (this is not true for operators on \mathcal{H}_E , though). In this section, we indicate which quantities remain meaningful in the thermodynamic limit and we give their precise definition. Note that the lattices $\mathbb{X}_n, \mathbb{A}_n$ remain meaningful with $L = \infty$, as was already indicated in Section 3.3. However, the spaces $\mathcal{R}_{n,A}$, defined as tensor products in finite volume, are no longer uniquely defined and we will choose a norm to specify them in Section 5.2.

5.1 Thermodynamic limits

Let us now indicate explicitly the volume dependence in the evolution operators $T_n^\Lambda, T_n^\Lambda(0)$, the correlation functions $G_{n,A}^\Lambda, G_{n,A}^{c,\Lambda}$ and the 'unnormalized states' $\nu_n^{\beta,\Lambda}, \nu_n^{\text{ref},\Lambda}$. Those objects were previously simply denoted by the same symbols without Λ but now we reserve this notation for infinite-volume quantities. The thermodynamic limit is then defined by point-wise convergence of kernels (well-defined since \mathbb{A}_n^Λ is naturally a subset of \mathbb{A}_n) Recall the assumption (2.17), it leads to

Lemma 5.1. *The kernels of the operators $T_n^\Lambda, T_n^\Lambda(0)$, the correlation functions $G_{n,A}^\Lambda, G_{n,A}^{c,\Lambda}$ and the 'unnormalized states' $\nu_n^{\beta,\Lambda}, \nu_n^{\text{ref},\Lambda}$ converge pointwise as $\Lambda \nearrow \mathbb{Z}^d$. The limiting kernels satisfy the recursion relations (3.31), (3.34) and the Ward identities (4.3) and (4.31). In particular, the sums on \mathbb{A} implicit in the RHS of these recursion relations are absolutely convergent.*

This lemma follows directly from the bounds of Lemma 10.1 and (for $\tau = 0$) those of Section 10.3.

5.2 Norm on kernels

5.2.1 Tensor product completions

We have introduced the space \mathcal{R}_n (on scale n) for operators acting on (rescaled) particle density matrices. As long as the volume Λ is finite, this space is finite-dimensional and hence isomorphic to $\mathcal{B}(\mathcal{B}_1(\mathcal{H}_S))$. Similarly, the tensor product spaces $\mathcal{R}_{A,n}$ are finite dimensional and thus all norms are equivalent. However, as $\Lambda \nearrow \mathbb{Z}^d$, care is needed. Even if we choose a norm for \mathcal{R}_n there are several ways to complete the algebraic tensor products. Our choice is dictated by the recursion relation (3.34) which we want to be continuous in the norm. The norms are of course meaningful for Λ finite as well, and in the definition below we will not distinguish different values of Λ .

Recall that an element of $\mathcal{R}_{n,A}$ is represented by the kernel $K_A = K_A(z'_A, z_A)$ with $A = \{\tau_1, \dots, \tau_m\}, \tau_i < \tau_{i+1}$ and $z_A = (z_{\tau_1}, \dots, z_{\tau_m}) \in \mathbb{A}_{n,A}$. Recall also that $z_\tau = (x_\tau, s_\tau)$. Since the scale subscript n does not play any role as far as the definition of the norm is concerned, we suppress it in what follows and we will simply write $\mathcal{R}, \mathcal{R}_A$, etc...

We need to distinguish in the norm the x and the s variables and so will view \mathcal{R} as the tensor product $\mathcal{R}^x \otimes \mathcal{R}^s$ where $K \in \mathcal{R}^x$ is represented by the kernel $K(x', x)$ and $K \in \mathcal{R}^s$ by $K(s', s)$. Introduce the following $L^1 - L^\infty$ norm in the spaces \mathcal{R}^x and \mathcal{R}^s

$$\|K\| := \max \left(\sup_{y'} \int dy (|K(y', y)|), \sup_y \int dy' |K(y', y)| \right) \quad (5.1)$$

where y stands either for x if $K \in \mathcal{R}^x$ and for s if $K \in \mathcal{R}^s$. The space \mathcal{R}_A can be viewed as the tensor product $\mathcal{R}_{\tau_m}^x \otimes \mathcal{R}_{\tau_m}^s \otimes \dots \otimes \mathcal{R}_{\tau_1}^x \otimes \mathcal{R}_{\tau_1}^s$. It is however simpler to consider more generally kernels $K \in \otimes_{i=1}^m \mathcal{R}^{\sigma_i}$ where $\sigma_i \in \{x, s\}$, hence not necessarily with the same number of x - and s -coordinates (Note that we drop the τ -labels since what follows is independent of the labelling of tensors). We now define the norm $\|K\|$ on such kernels, inductively in the degree m .

Write $K = K(\underline{y}', \underline{y})$ with $\underline{y} = (y_1, \dots, y_m)$ and y_i equals x_i or s_i depending on σ_i , and analogously for \underline{y}' . For $1 \leq i \leq m$, fix $y_{<} = (y_1, \dots, y_{i-1})$ and $y_{>} = (y_{i+1}, \dots, y_m)$ and let $K^i \in \mathcal{R}^{\sigma_i}$ be the restriction of K to \mathcal{R}^{σ_i} with

$y_{<}$ and $y_{>}$ kept fixed. Then $K_i := \|K^i\|$ is a kernel of degree $m - 1$. We define the norm inductively in the degree of kernels by setting

$$\|K\| := \max_i \|K_i\|.$$

The virtue of this norm is that it behaves well under tensor products and contractions, the basic building blocks of the recursion relation (3.34). The contraction analogous⁴ to (3.3) $\iota : \mathcal{R}_n^\sigma \otimes \mathcal{R}_n^\sigma \rightarrow \mathcal{R}_n^\sigma$ is given by

$$(\iota K)(y', y) = \int d\tilde{y} K(\tilde{y}, y'; y, \tilde{y}) \quad (5.2)$$

We extend this map to $\otimes_{i=1}^m \mathcal{R}^{\sigma_i}$ in the natural way: for i, j s.t. $\sigma_i = \sigma_j$ let ι_{ij} act in the i :th and the j :th factors $\mathcal{R}^{\sigma_j} \otimes \mathcal{R}^{\sigma_i}$. Finally, we write $|K|$ for the operator with kernel $|K|(y', y) := |K(y', y)|$. Points 1) and 2) of the following lemma are used throughout, point 3) is necessary only in the proof of Lemma 7.2.

Lemma 5.2. *Let K and L be kernels as above. Then*

1)

$$\|K \otimes L\| \leq \|K\| \|L\| \quad (5.3)$$

2) *Let $\sigma_i = \sigma_j$ such that ι_{ij} is defined, then*

$$\|\iota_{ij} K\| \leq \|K\| \quad (5.4)$$

3) *Let K be of degree m with $\sigma_1 = \sigma_2 = \dots = \sigma_m$, then, for any $i \in \{1, \dots, m\}$,*

$$\sup_{y, y'} |\iota_{12} \iota_{23} \dots \iota_{m-1, m} K| \leq \|\sup_{y_i, y'_i} |K|\| \quad (5.5)$$

where on the RHS the norm is applied to a kernel of degree $m - 1$.

Proof. For the first claim, we proceed by induction in the degree of $K \otimes L$. Obviously $(K \otimes L)_i = K^j \otimes |L|$ if $i = \text{degree}(L) + j$, and $(K \otimes L)_i = |K| \otimes L_i$ if $i \leq \text{degree}(L)$. Thus

$$\|K \otimes L\| \leq \max_{ij} (\|K_i \otimes L\|, \|K \otimes L_j\|) \leq \max_{ij} (\|K_i\| \|L\|, \|K\| \|L_j\|) = \|K\| \|L\|$$

where we used induction in the second step.

For the second claim, the inductive definition of our norm means it suffices to check the claim for K of degree two. Thus consider the first term in (5.1) for ιK :

$$\sup_{y'} \int dy \left| \int d\tilde{y} K(\tilde{y}, y'; y, \tilde{y}) \right| \leq \sup_{y'} \int d\tilde{y} \int dy |K(\tilde{y}, y'; y, \tilde{y})| \leq \sup_{y'} \int d\tilde{y} \sup_{y''} \int dy |K(y'', y'; y, \tilde{y})|$$

and the expression on the right is bounded by $\|K_1\|$. By analogous reasoning, the second term in (5.1) is bounded by $\|K_2\|$ and hence we have indeed $\|\iota K\| \leq \|K\|$.

To get the third claim, it is sufficient to check it for $m = 2$ and for $m = 3$ with $i = 2$, since all other cases can be reduced to one of these. We check explicitly the case $m = 3$ with $i = 2$. The LHS is estimated as

$$\begin{aligned} \sup_{y, y'} \int d\tilde{y} \int d\tilde{y}' |K(\tilde{y}, \tilde{y}', y'; y, \tilde{y}, \tilde{y}')| &\leq \sup_{y, y'} \int dy''' \sup_{\tilde{y}} \int d\tilde{y} \sup_{y''} |K(y''', y'', y'; y, \tilde{y}, \tilde{y}')| \\ &\leq \sup_y \int dy''' \sup_{y'} \int d\tilde{y} \sup_{\tilde{y}, y''} |K(y''', y'', y'; y, \tilde{y}, \tilde{y}')| \end{aligned}$$

and one now verifies easily that the result is indeed smaller than the RHS. The $m = 2$ case follows by similar, though simpler reasoning. \square

⁴The contraction defined in (3.3) acts in both x and s coordinates, whereas the one defined in (5.2) acts in one of them

5.2.2 Definition of the norm $\|\cdot\|_\gamma$

We will modify the norms introduced above to encode information about the decay properties of the kernels. Thus we will define a norm $\|K_A\|_\gamma$ that depends on two parameters: γ_0 (giving the decay in the $v - v'$ coordinate, and γ given the decay in the $x - x'$ coordinate. Note that the parameter γ_0 is not included in the notation for the norm, since we will never need to consider a different one. The parameter γ will equal a multiple (of order 1) of γ_0 . Given a kernel $K_A \in \mathcal{R}_A$, let

$$K_A^\gamma(z'_A, z_A) := K_A(z'_A, z_A) e^{\gamma \sum_{\tau \in A} |x'_\tau - x_\tau|} e^{\gamma_0 \sum_{\tau \in A} |v'_\tau - v_\tau|} \quad (5.6)$$

where we wrote $z_\tau = (x_\tau, s_\tau)$ and $s_\tau = (v_\tau, \eta_\tau, (e_L)_\tau, (e_R)_\tau)$. Then we set

$$\|K_A\|_\gamma := \|K_A^\gamma\|. \quad (5.7)$$

Note that

$$(K_A \otimes K_B)^\gamma = K_A^\gamma \otimes K_B^\gamma. \quad (5.8)$$

Using triangle inequality $|y' - y| \leq |y' - \tilde{y}| + |\tilde{y} - y|$ in (5.2) we get for point-wise *nonnegative* kernels K_A

$$(\iota_{\tau\tau'} K_A)^\gamma \leq \iota_{\tau\tau'} K_A^\gamma. \quad (5.9)$$

with the contraction ι acting in both x and s coordinates (as defined in (3.3)). These observations lead to

Lemma 5.3. 1) *Let \mathcal{A} be a collection of disjoint sets $A \subset \mathbb{N}_0$ such that $I_{A'} = \text{Supp } A$ (see Section 3.5) and let for all $A \in \mathcal{A}$ an operator $K_A \in \mathcal{R}_A$ be given, then*

$$\left\| \mathcal{T}_{A'} \left[\bigotimes_{A \in \mathcal{A}} K_A \right] \right\|_\gamma \leq \prod_{A \in \mathcal{A}} \|K_A\|_\gamma \quad (5.10)$$

2) *Let $K_A \in \mathcal{R}_A$. Then*

$$\|\mathbf{S}_\ell[K_A]\|_\gamma = \|K_A\|_{\gamma/\ell}, \quad (5.11)$$

and

$$\|K_A\|_\gamma \leq \|K_A\|_{\gamma'}$$

for $\gamma < \gamma'$.

Proof. For 1) we bound (5.10) from above by replacing K_A by $|K_A|$ and then use (5.8) and (5.9) and the fact that \mathcal{T} is a product of contractions ι_{ij} . For the first claim in 2) observe that the norm (5.1) is invariant under \mathbf{S}_ℓ and the effect in γ is the stated one. The second claim is obvious. \square

Remark From now on, the space \mathcal{R} is meant to be equipped with the norm $\|\cdot\|_\gamma$. The exponential factors in our norm (provided $\gamma > 0$) imply in particular that kernels $K \in \mathcal{R}$ map the space

$$\mathcal{B}_p(\mathcal{H}_S) = \left\{ V \in \mathcal{B}(\mathcal{H}_S) \mid \text{Tr}(VV^*)^{p/2} < \infty \right\}$$

into itself, for any $1 \leq p < \infty$.

5.2.3 Definition of the norm $\|\cdot\|_\mathcal{G}$

When studying phenomena like diffusion that are intimately connected with the translation invariance of the system, we will mainly need to consider the coordinates x, x' (see Section 5.3.1 on translation invariance). Therefore, we often want to consider $K \in \mathcal{R} = \mathcal{R}^x \otimes \mathcal{R}^s$ as a kernel in x, x' but taking values in the 'internal space' \mathcal{R}^s which for short will be called \mathcal{G} . \mathcal{G} is completed with the norm

$$\|F\|_\mathcal{G} := \max \left(\sup_{s'} \sum_s |F(s', s)| e^{\gamma_0 |v - v'|}, \sup_s \sum_{s'} |F(s', s)| e^{\gamma_0 |v - v'|} \right) \quad (5.12)$$

Obviously, the norm $\|\cdot\|_{\mathcal{G}}$ relates well to $\|\cdot\|_{\gamma}$ on \mathcal{R} . Given $K \in \mathcal{R}$,

$$\max \left(\sup_{x'} \int dx \|K(x', x)\|_{\mathcal{G}} e^{\gamma|x-x'|}, \sup_x \int dx' \|K(x', x)\|_{\mathcal{G}} e^{\gamma|x-x'|} \right) \leq \|K\|_{\gamma}. \quad (5.13)$$

Indeed, looking back at Section 5.2.1 and taking $\sigma_1 = x$ and $\sigma_2 = s$, the LHS equals $\|K_2^{\gamma}\|$ whereas the RHS equals $\max_{i=1,2} \|K_i^{\gamma}\|$.

5.3 Symmetries

5.3.1 Translation invariance

We implement spatial translations on $\mathcal{H}_{\mathbb{S}} = \mathcal{S} \otimes l^2(\mathbb{Z}^d)$ by the operators $V_u \in \mathcal{B}(\mathcal{H}_{\mathbb{S}})$, $u \in \mathbb{Z}^d$;

$$(V_u \psi)(y) := \psi(y + u) \in \mathcal{S}, \quad \psi \in \mathcal{H}_{\mathbb{S}} \quad (5.14)$$

Write in general $\text{Ad}(S)O = SOS^{-1}$ for $O, S \in \mathcal{B}(\mathcal{H}_{\mathbb{S}})$. Then $\text{Ad}(V_u)$ implements translations on density matrices. In our \mathbb{A}_0 -coordinates, it acts as

$$\text{Ad}(V_u)\rho(x, s) = \rho(x + u, s) \quad (5.15)$$

We will also denote by $\text{Ad}(V_u)$ the operator on \mathbb{A}_n defined by (5.15), now with $x, u \in \mathbb{X}_n$. It is then natural to call a $K \in \mathcal{R} = \mathcal{R}_n$ translation invariant if it commutes with translations, i.e.

$$V_u K V_{-u} = K, \quad \text{for } u \in \mathbb{X}_n \quad (5.16)$$

For such a translation invariant operator K , we will often abbreviate the reduced kernel

$$K(x' - x) = K(x', x) \quad (K(x', x) \in \mathcal{G}) \quad (5.17)$$

and we have

Lemma 5.4. *Let $K \in \mathcal{R}_n$ be translation-invariant, then*

$$\|K\|_{\gamma} = \int_{\mathbb{X}_n} dx \|K(x)\|_{\mathcal{G}} e^{\gamma|x|} \quad (5.18)$$

Proof. The inequality \geq follows from (5.13). Following the discussion after (5.13), in order to get \leq , we need to bound $\|K_1^{\gamma}\|$ by $\|K_2^{\gamma}\|$. This follows since the supremum over the x or x' coordinate can be dropped because of translation invariance and hence we get an upper bound by moving the supremum over the s or s' coordinate to the right. \square

5.3.2 Fourier Transform

The translation invariance suggests to Fourier transform the kernel in the variable $x \in \mathbb{X}_n$. The dual space to the lattice \mathbb{X}_n is the torus

$$\mathbb{T}_n := (\ell^n \mathbb{T})^d \quad (5.19)$$

We define for $p \in \mathbb{T}_n$,

$$\hat{\rho}(p, s) := \int_{\mathbb{X}_n} dx e^{ipx} \rho(x, s), \quad \text{for } \rho \in \ell^1(\mathbb{A}_n), \quad (5.20)$$

and, for $K \in \mathcal{R}_n$,

$$\hat{K}(p) := \int_{\mathbb{X}_n} dx e^{ipx} K(x), \quad (\hat{K}(p) \in \mathcal{G}). \quad (5.21)$$

where the sum on the RHS is absolutely convergent if $\|K\|_{\gamma}$ is finite for some $\gamma \geq 0$. It follows that, if we have two translation invariant kernels K_1, K_2 taking values in \mathcal{G} , then

$$\widehat{K_1 K_2}(p) = \hat{K}_1(p) \hat{K}_2(p) \quad (5.22)$$

where the product on the LHS is in \mathcal{R}_n and on the RHS in \mathcal{G} . The following standard consequence of exponential decay will be used throughout. If K is a translation-invariant kernel then

$$\sup_{\text{Im } p \leq \gamma} \|\hat{K}(p)\|_{\mathcal{G}} \leq \|K\|_{\gamma} \quad (5.23)$$

5.3.3 Symmetries of the lattice

Let us investigate the action of a lattice symmetry O on our operators. We consider a density matrix ρ , and we abbreviate the transformed density matrix as $\rho_O \equiv \text{Ad}(V_O)\rho$, such that

$$\rho_O(x_L, x_R) = \rho(Ox_L, Ox_R) \quad (5.24)$$

To write this transformation in our new coordinates x, s , we let x_O, v_O, η_O be the coordinates corresponding to Ox_L, Ox_R (we suppress the coordinates e_L, e_R since they are untouched by O). Define the vector \hat{e} by

$$\hat{e}_j = \begin{cases} 0 & \text{if } (O\eta)_j \in \{0, 1\} \\ 1 & \text{if } (O\eta)_j = -1 \end{cases} \quad (5.25)$$

Then, one checks that

$$x_O = Ox - \hat{e}, \quad v_O = Ov - \hat{e}, \quad \eta_O = O\eta + 2\hat{e} \quad (5.26)$$

Hence $\rho_O(x, v, \eta) = \rho(Ox - \hat{e}, Ov - \hat{e}, O\eta + 2\hat{e})$. We compute the fourier transform of ρ in the variable x .

$$\hat{\rho}_O(p, v, \eta) = e^{ip\hat{e}} \hat{\rho}(O^{-1}p, Ov - \hat{e}, O\eta + 2\hat{e}) = (I_{p,O} \hat{\rho}(O^{-1}p, \cdot, \cdot)) (v, \eta) \quad (5.27)$$

where $I_{p,O}$ is an invertible transformation acting on the degrees of freedom v, η .

6 Convergence to a fixed point

We now state the induction hypotheses for the RG flow of the reduced dynamics T_n (Proposition 6.1), and the correlation functions $G_{n,A}^c$ (Proposition 6.2). Their validity for all n implies our main results, as we show below in Section 6.2. The inductive proof of these induction hypotheses is postponed to Sections 7-11 and we provide a brief guide to the proof in Section 6.1.3.

6.1 Induction hypotheses

Let us first discuss constants and small parameters of our theory. The Hamiltonians H_S, H_E, H_I as well as the environment correlation function ζ and temperatures β, β_i are considered fixed and constants depending only on them are denoted by C (large constants) and c (small constants). The adjustable parameter in the problem is the coupling λ and in the proof also the RG scaling factor ℓ and τ_0 entering the initial time scale $t_0 = \lambda^{-2}\tau_0$.

We define the exponents

$$\tilde{\alpha} := \begin{cases} (2\alpha - 1)/4 & 1/2 < \alpha < 1 \\ (4\alpha - 1)/8 & 1/4 < \alpha \leq 1/2 \end{cases}, \quad \tilde{\alpha}_I := \begin{cases} \text{not defined} & 1/2 < \alpha < 1 \\ \alpha/4 & 1/4 < \alpha \leq 1/2 \end{cases} \quad (6.1)$$

with α the correlation decay exponent in Assumption A. The exponent α_I (I for 'initial') appears in the treatment of correlation functions G_A^c with $A \ni 0$ ('boundary' correlation functions; those were defined only in the case $\beta_1 = \beta_2$) and we define this exponent only for $\alpha \leq 1/2$. Then, we introduce the 'running coupling constants'

$$\epsilon_n := \ell^{-n\tilde{\alpha}} \epsilon_0, \quad \epsilon_0 := C|\lambda|^\alpha \quad (6.2)$$

$$\epsilon_{I,n} = \ell^{-n\tilde{\alpha}_I} \epsilon_{I,0}, \quad \epsilon_{I,0} = C|\lambda|^{2-2\alpha} \quad (6.3)$$

In the Sections 7, 8 and 9 the λ appears only through $\epsilon_0, \epsilon_{1,0}$ which can be considered the fundamental small parameters.

Our basic convention is that ℓ is chosen large enough compared to the fixed parameters and we will for example freely assume that $\ell^{-1}C < 1$. The coupling constant $|\lambda|$ (or $\epsilon_0, \epsilon_{1,0}$) is then chosen small compared to ℓ , such that we can freely assume that $|\lambda|C(\ell) < 1$ for quantities $C(\ell)$ depending on ℓ but not on λ . Finally, unless otherwise stated all constants are uniform in n , the RG iteration.

6.1.1 Induction hypothesis for T_n

The first induction hypothesis concerns the reduced evolution T_n and $T_n(\tau = 0)$ of (4.20). The hypothesis involves parameters \mathbf{p}_0, γ_0 and D_0 that will be fixed in Section 11; they result from the weak coupling analysis.

Proposition 6.1 (Induction hypothesis for T_n). *There exist $\tau_0, \ell_0 < \infty$ and $\lambda(\ell) > 0$ such that for $\ell > \ell_0$ and $|\lambda| \leq \lambda(\ell)$ the following holds, uniformly in $\mathbf{t}_0 \in [\tau_0, 2\tau_0]$ and in n .*

- 1) **Analyticity.** *The operators $\hat{T}_n(p)$ and $\hat{T}_n(0, p)$ (\mathbf{t}_0 refers to $\tau = 0$) are analytic in a strip of width γ_0 with*

$$\sup_{|\operatorname{Im} p| \leq \gamma_0} \|\hat{T}_n(p)\|_{\mathcal{G}} \leq C, \quad \sup_{|\operatorname{Im} p| \leq \gamma_0} \|\hat{T}_n(0, p)\|_{\mathcal{G}} \leq C \quad (6.4)$$

Moreover, $\hat{T}_n(0, p) = \hat{T}_{n-1}(0, p/\ell)$ for $n \geq 1$.

- 2) **Diffusion.** *Let $|\operatorname{Re} p| < \mathbf{p}_n$ given below and $|\operatorname{Im} p| \leq \gamma_0$. The operators $\hat{T}_n(p)$ have a simple eigenvalue⁵ $e^{f_n(p)}$, i.e.*

$$R_n(p)\hat{T}_n(p) = e^{f_n(p)}R_n(p) \quad (6.5)$$

where the one-dimensional spectral projector $R_n(p)$ is bounded as

$$\|R_n(p)\|_{\mathcal{G}} \leq C. \quad (6.6)$$

and the complementary part as

$$\|(1 - R_n(p))\hat{T}_n(p)\|_{\mathcal{G}} \leq \frac{1}{2}\ell^{-\frac{\tilde{\alpha}}{8}n} =: \mathbf{b}_n, \quad (6.7)$$

The analytic function $f_n(p)$ satisfies

$$|f_n(p) + D_n p^2| < \ell^{-\frac{\tilde{\alpha}}{4}n} |p|^3 \quad (6.8)$$

Moreover, for $n \geq 1$,

$$\|R_n(p) - R_{n-1}(p/\ell)\|_{\mathcal{G}} \leq \sqrt{\epsilon_0} \ell^{-\frac{\tilde{\alpha}}{4}(n-1)}, \quad |D_n - D_{n-1}| \leq \sqrt{\epsilon_0} \ell^{-\frac{\tilde{\alpha}}{4}(n-1)} \quad (6.9)$$

The constant \mathbf{p}_n is given by

$$e^{-\frac{1}{2}D_n \mathbf{p}_n^2} = \ell^{-\frac{\tilde{\alpha}}{4}n} e^{-\frac{1}{2}D_0 \mathbf{p}_0^2}. \quad (6.10)$$

and $\mathbf{p}_0 \leq D_0/2$. The spectral projection $R_n(0)$ is given by

$$R_n(0) = |\mu_{T_n}\rangle \langle 1_{\mathcal{S}_0}| \quad (6.11)$$

(notation as in Section 4.1) where μ_{T_n} defines a probability measure on \mathcal{S}_0 : $\sum_{s \in \mathcal{S}_0} \mu_{T_n}(s) = 1$ and $\mu_{T_n}(s) \geq 0$. Moreover in the equilibrium case $\beta_1 = \beta_2 = \beta$

$$\|R_n(0) - |\mu^\beta\rangle \langle 1_{\mathcal{S}_0}|\|_{\mathcal{G}} \leq C\sqrt{\epsilon_0} \quad (6.12)$$

with μ^β as in Section 4.2.1.

⁵For example, consider $\hat{T}_n(p)$ as an operator on $l^\infty(\mathcal{S})$, see the discussion in Appendix A

3) **Gap.** Let $|\operatorname{Re} p| \geq \mathfrak{p}_n$ and $|\operatorname{Im} p| \leq \gamma_0$. Then

$$\|\hat{T}_n(p)\|_{\mathcal{G}} \leq \mathfrak{b}_n \quad (6.13)$$

4) In position space we have

$$\|T_n(x)\|_{\mathcal{G}} \leq C e^{-\gamma_0|x|} \quad (6.14)$$

For later use we note that (6.8) together with $\mathfrak{p}_0 \leq D_0/2$ implies

$$e^{-\frac{3}{2}D_n(\operatorname{Re} p)^2 - C(\operatorname{Im} p)^2} \leq |e^{f_n(p)}| \leq e^{-\frac{1}{2}D_n(\operatorname{Re} p)^2 + C(\operatorname{Im} p)^2}. \quad (6.15)$$

6.1.2 Induction hypothesis for $G_{n,A}^c$

We move to the correlation functions $G_{n,A}^c$. First, we define a distance-like function on sets A of times as follows: assume that $A = \{\tau_1, \dots, \tau_m\}$ with $\tau_1 < \dots < \tau_m$, then

$$\operatorname{dist}(A) = \operatorname{dist}(\tau_1, \dots, \tau_m) := \prod_{j=1}^{m-1} (1 + |\tau_{j+1} - \tau_j|) \quad (6.16)$$

and we will always use $\operatorname{dist}(A)^\alpha = (\operatorname{dist}(A))^\alpha$ to quantify the decay in time of the operators $G_{n,A}^c$, with α as in Assumption A. Since $G_{n,A}^c$ are translation invariant in time if $0 \notin A$ it suffices to consider two cases: $\min A = 0$ and $\min A = 1$.

Proposition 6.2 (Induction hypothesis for G_A^c). *Let $\mathfrak{t}_0, \ell, \lambda$ be as in Proposition 6.1 and recall ϵ_n defined in (6.1). (a) Let $1 \geq \alpha > 1/2$. Then*

$$\sum_{A \subset \mathbb{N}; |A|=k, \min A=1} \operatorname{dist}(A)^\alpha \|G_{n,A}^c\|_{10\gamma_0} \leq \epsilon_n^k, \quad k \geq 2 \quad (6.17)$$

(b) If the environment is in equilibrium, i.e., if $\beta_1 = \beta_2$, then (a) holds for $1/4 < \alpha \leq 1/2$. In that case the correlation functions with $A \ni 0$ satisfy

$$\sum_{A \subset \mathbb{N}; |A|=k, \min A=0} \operatorname{dist}(A)^\alpha \|G_{n,A}^c\|_{10\gamma_0} \leq \epsilon_{1,n} \epsilon_n^k, \quad k \geq 2 \quad (6.18)$$

6.1.3 Plan of proof of induction hypotheses

Given the induction hypotheses Proposition 6.1 and Proposition 6.2 up to scale n , we prove Proposition 6.1 on scale $n+1$ in Section 7. This result is stated explicitly in Section 7.3.

The proof of Proposition 6.2 on scale $n+1$ is spread over Sections 8 and 9. In the former section, we treat the linear part of the recursion relation and in the latter the nonlinear part (notions not yet defined). The nonlinear part is straightforward and one does not need to distinguish between the cases $\alpha > 1/2$ and $\alpha \leq 1/2$. In particular, the equilibrium condition $\beta_1 = \beta_2$ plays no role here. The linear part is slightly tricky; to treat the case $\alpha \leq 1/2$, we have to exploit the equilibrium condition and consider correlation functions with $A \ni 0$. The linear part also dictates the choice of the exponents $\tilde{\alpha}, \tilde{\alpha}_I$ in eqs. (6.2,6.3), respectively.

Then, we need to establish the Induction hypotheses on scale $n=0$. In Section 10, we establish Proposition 6.2 and in Section 11, we establish Proposition 6.1. In those sections, we also outline how to choose the constants γ_0, \mathfrak{p}_0 and D_0 .

6.2 Proof of the main theorems

We assume the Induction hypotheses Proposition 6.1 and Proposition 6.2 for all n . Then we have

Proposition 6.3. *Let $\mathfrak{t}_0, \ell, \lambda$ be as in Proposition 6.1. There is a projector $P^\star \in \mathcal{G}$ of the form $|\mu^\star\rangle\langle 1_{\mathcal{S}_0}|$, and a diffusion constant $\tilde{D}^\star > 0$ such that for p in the strip $|\operatorname{Im} p| < \gamma_0$*

$$\left\| \hat{T}_n(p) - e^{-\tilde{D}^\star p^2} P^\star \right\|_{\mathcal{G}} \leq C \ell^{-cn} \quad (6.19)$$

for some exponent $c > 0$. In the case $\beta_1 = \beta_2$, we have $\mu^\star = \mu^\beta$, the projected Gibbs state (4.11).

Proof. Let first $|\operatorname{Re} p| \geq \mathfrak{p}_n$. Then the claim follows from the definition (6.10) and the bounds (6.13) and (6.15). For $|\operatorname{Re} p| \leq \mathfrak{p}_n$ note first that by (6.9) both $R_n(0)$ and D_n are convergent sequences. Calling their limits P^\star, \tilde{D}^\star respectively we get

$$\|R_n(0) - P^\star\|_{\mathcal{G}}, |D_n - \tilde{D}^\star| \leq \epsilon_n^{1/4}. \quad (6.20)$$

Let $m = cn$ where $0 < c < 1$ is such that $|\ell^{-m/2} p| \leq \gamma_0$. Such a c exists since $\mathfrak{p}_n \geq |\operatorname{Re} p|$ and \mathfrak{p}_n grows not faster than $(Cn \log \ell)^{1/2}$ with n . Write

$$\begin{aligned} \|R_n(0) - R_n(p)\|_{\mathcal{G}} &\leq \sum_{j=0}^{m-1} \|R_{n-(j+1)}(p\ell^{-(j+1)}) - R_{n-j}(p\ell^{-j})\|_{\mathcal{G}} \\ &+ \|R_n(0) - R_{n-m}(0)\|_{\mathcal{G}} + \|R_{n-m}(p\ell^{-m}) - R_{n-m}(0)\|_{\mathcal{G}} \end{aligned} \quad (6.21)$$

The first and second term on the RHS is bounded by $\epsilon_{n-m}^{1/4} = \epsilon_{(1-c)n}^{1/4}$ using (6.9). The third term is bounded by $C(\gamma_0)^{-1} \ell^{-m/2}$ by analyticity of $p \rightarrow R_{n-m}(p)$ in the ball of radius γ_0 at origin. Hence

$$\|R_n(0) - R_n(p)\|_{\mathcal{G}} \leq C \ell^{-cn}. \quad (6.22)$$

Next, bound

$$|e^{f_n(p)} - e^{-\tilde{D}^\star p^2}| \leq |e^{f_n(p)} - e^{-D_n p^2}| + |e^{-D_n p^2} - e^{-\tilde{D}^\star p^2}| \quad (6.23)$$

$$\leq C(|f_n(p) + D_n p^2| + |D_n - \tilde{D}^\star|) \leq C \ell^{-cn} \quad (6.24)$$

by (6.8) and (6.20), and the slow growth of \mathfrak{p}_n .

Combining (6.20), (6.22) and (6.24) with (6.7) yields the bound (6.19). It remains to argue that $P^\star = P^\beta$ in the case $\beta_1 = \beta_2 = \beta$. Take Λ finite and recall that $\rho_{\text{SE}}^\beta = \mathcal{U}_{n=0}(\tau=0)\rho_{\text{SE}}^{\text{ref}}$ and, by the invariance of the Gibbs state

$$\rho_S^\beta = \operatorname{Tr}_E[e^{-itL}\rho_{\text{SE}}^\beta] = \check{Z}_t \rho_S^{\text{ref}}, \quad t \geq 0 \quad (6.25)$$

with \check{Z}_t defined in Section 4.2.2. On the other hand,

$$\mathbf{S}_{\ell^n}[\check{Z}_{\ell^{2n}t_0}] = \mathbb{E}(\mathcal{U}_n \mathcal{U}_n(0)) = T_n T_n(0) + \mathcal{T}[G_{\{0,1\},n}^c] \quad (6.26)$$

where (0) refers to $\tau = 0$. On both sides, we can take the thermodynamic limit. Moreover, we know that $\|G_{\{0,1\},n}^c\|_{10\gamma_0} \leq e^{-cn}$ by Proposition 6.2. Let us then Fourier transform (6.26) and apply the $p = 0$ -component to μ^{ref} , this yields

$$\mu^\beta = \hat{T}_n(0) \hat{T}_n(0,0) \mu^{\text{ref}} + \mathcal{O}(e^{-cn}) \quad (6.27)$$

where the LHS follows from (6.25) and we wrote $\hat{T}_n(0,0) = \hat{T}_n(p=0, \tau=0)$, $\hat{T}_n(0) = \hat{T}_n(p=0)$. In fact, by (6.25) for $t = 0$, we have also $\hat{T}_n(0,0) \mu^{\text{ref}} = \mu^\beta$ and by Proposition 6.3, we have $\hat{T}_n(0) = P^\star + \mathcal{O}(e^{-cn})$. Hence

$$\mu^\beta = P^\star \mu^\beta \quad (6.28)$$

and this of course implies $P^\star = P^\beta$. □

6.2.1 Proof of results in Section 2.4 along a subsequence of times

We argue that Proposition 6.3 implies the results of Section 2.4 along the sequence of times $t_n := \ell^{2n} t_0$. The resulting diffusion constant is given by

$$D^* = t_0^{-1} \tilde{D}^* = \mathbf{t}_0^{-1} \lambda^2 \tilde{D}^* \quad (6.29)$$

Let us first write the claims of Theorems 1 and 2 in terms of the RG. Express the time t in units of the kinetic time scale, $t = st_0$, $t_0 = \lambda^{-2} \mathbf{t}_0$. Then we have

$$\mathrm{Tr}[e^{ip \frac{X}{\sqrt{t}}} \rho_{S,t}] = \mathrm{Tr}[e^{ip \frac{X}{\sqrt{t_0}}} \mathbf{S}_{\sqrt{s}}[Z_{st_0}] S_{\sqrt{s}} \rho_{S,0}] \quad (6.30)$$

For $t = t_n = \ell^{2n} t_0$, $\mathbf{S}_{\sqrt{s}}[Z_{st_0}] = T_n$ and by the Fourier transform, we get

$$\mathrm{Tr}[e^{ip \frac{X}{\sqrt{t_n}}} \rho_{S,t_n}] = \mathrm{tr}[\hat{T}_n(\frac{p}{\sqrt{t_0}}) \hat{\rho}_{S,0}(\frac{p}{\ell^n \sqrt{t_0}})] \quad (6.31)$$

where the 'trace' tr is defined by $\mathrm{tr}[\psi] = \sum_{s \in S_0} \psi(s)$. Both \hat{T}_n (by Proposition 6.3) and $\hat{\rho}_{S,0}$ (by the finite-range condition (2.11)) are analytic in p and uniformly bounded in the strip $|\mathrm{Im} p| < \gamma_0$. Hence, by Proposition 6.3 (6.31) converges as $n \rightarrow \infty$ to

$$e^{-D^* p^2} \mathrm{tr}[P^* \hat{\rho}_{S,0}(0)] = e^{-D^* p^2}.$$

By the Vitali theorem, the derivatives converge as well and Theorem 1 and 2 follow (along a subsequence).

For Theorems 3 and 4 we need to consider translation invariant observables A instead of $e^{ip \frac{X}{\sqrt{t}}}$ in (6.31). Its kernel $A(x, s) = A(s)$ is constant in x and so we get from Proposition 6.3

$$\lim_{n \rightarrow \infty} \mathrm{Tr} A \rho_{S, \ell^{2n}} = \mathrm{tr}[A P^* \hat{\rho}_{S,0}(0)] = \mathrm{tr}[A \mu^*] \quad (6.32)$$

In the case where $\mu^* = \mu^\beta$, the last line of course equals $\langle A \rangle_\beta$. The decoherence result (2.35) is a simple consequence of the fact that $\|P^*\|_{\mathcal{G}} < C$, hence $\sum_s \mu^*(s) e^{\gamma_0 |v|} < C$.

6.2.2 Proof of results in Section 2.4 for general times

Let us now consider general times in (6.30). We will generalize a bit our RG iteration. It is easy to see that we can run the RG iteration i.e. Propositions 6.2 and 6.3 as well with scaling factor 2ℓ (by possibly reducing λ) and we can also at each iteration step choose arbitrarily between the two factors. Let $\sigma \in \{0, 1\}^{\mathbb{N}}$ label the possible choices: if $\sigma_n = 0$ at the n :th step apply scaling ℓ , if $\sigma_n = 1$ apply scaling 2ℓ . Denote the resulting sequence of operators $T_{n,\sigma}$. They satisfy Proposition 6.3 uniformly in σ and hence we obtain projectors P_σ^* and diffusion constants D_σ^* .

Lemma 6.1. P_σ^*, D_σ^* are independent of σ .

Proof. Denote the corresponding times by $t_{n,\sigma}$:

$$t_{n,\sigma} = \lambda^{-2} \ell^{2n} 4^m \mathbf{t}_0 \quad (6.33)$$

where $m = m(n, \sigma) := \#\{i \leq n : \sigma_i = 1\}$. Since $T_{n,\sigma} = \mathbf{S}_{(t_{n,\sigma}/t_0)^{\frac{1}{2}}} Z_{t_{n,\sigma}}$ we have $T_{N,\sigma} = T_{N,\sigma'}$ whenever $t_{N,\sigma} = t_{N,\sigma'}$. In such case, since both sequences satisfy the bounds of Proposition 6.3 we conclude

$$\|P_\sigma^* - P_{\sigma'}^*\|_{\mathcal{G}}, |D_\sigma^* - D_{\sigma'}^*| \leq C \ell^{-cN}. \quad (6.34)$$

Now, given σ, σ' and n there exists $N \geq n$ and $\tilde{\sigma}, \tilde{\sigma}'$ s.t. $\tilde{\sigma}_i = \sigma_i$ for all $i \leq n$, similarly for the primes, and $t_{N,\tilde{\sigma}} = t_{N,\tilde{\sigma}'}$. Indeed, just take $N = n + |m(n, \sigma') - m(n, \sigma)|$ and $\tilde{\sigma}'_i = 0, \tilde{\sigma}'_i = 1$ for $i > n$ in case $m(n, \sigma') - m(n, \sigma) > 0$ and vice versa in the opposite case. Hence, we conclude that (6.34) holds for all σ, σ' and N and the claim follows. \square

Since the claims of the Propositions 6.2 and 6.3 are uniform in the initial time $\mathbf{t}_0 \in [\tau_0, 2\tau_0]$ as well we have σ -independent limits $P_{\sigma, \mathbf{t}_0}^* \equiv P_{\mathbf{t}_0}^*, D_{\sigma, \mathbf{t}_0}^* \equiv D_{\mathbf{t}_0}^*$. Let us also denote the \mathbf{t}_0 -dependence of the times (6.33) explicitly as $t_{n,\sigma,\mathbf{t}_0}$. Let $U \subset [\tau_0, 2\tau_0]$ be the set of \mathbf{t}_0 s.t. there exist σ, σ' and n, n' such that $t_{n,\sigma,\mathbf{t}_0} = t_{n',\sigma',\mathbf{t}_0}$. By the same reasoning as in the proof above, one deduces that on the set U , the limits $P_{\mathbf{t}_0}^*, D_{\mathbf{t}_0}^*$ are constant.

Lemma 6.2. *If $\log 2 / \log \ell$ is irrational, then U is dense in $[\tau_0, 2\tau_0]$.*

Proof. Consider $\log t_{n,\sigma,t_0} = -2 \log \lambda + \log t_0 + 2n \log \ell + 2m \log 2$. If $\log 2 / \log \ell$ is irrational, then the map $x \rightarrow x + \log 2 / \log \ell \pmod{1}$ has dense orbits and the claim follows easily from this. \square

Using a density of orbits argument as in the previous Lemma it is easy to see that there exists $T < \infty$ such that the set of times t_{n,σ,t_0} with $t_0 \in U$ is dense in $[T, \infty)$. Along any sequence of times from this dense set, the limit (2.29) exists and is independent of the chosen sequence. By the strong continuity of the Z_t (which will be easily derived in Lemma 10.1), the function (2.29) is continuous in t and hence the limit is independent of ℓ, t_0 . This completes the proof of Theorems 1 and 2. Theorem 3 follows analogously by considering sequences of times in (2.33).

7 Flow of T

As announced in Section 6.1.3, we prove the Induction step for T . The following convention applies to Sections 7, 8 and 9: We always assume that the induction hypotheses Propositions 6.1 and 6.2 are satisfied on scale n (we then prove them on scale $n+1$), and we do not repeat this assumption in the statements of all lemmata and propositions. Throughout this Section, as well as in Sections 8 and 9, we use the conventions on $\ell, \epsilon_0, \epsilon_{I,0}$ that were explained at the beginning of Section 6.1.

7.1 Powers of T_n

We show how to bound powers of T_n and we abbreviate $T = T_n$ whenever no confusion is possible. This bound will be an important ingredient of both induction steps for T_n and $G_{n,A}^c$.

Lemma 7.1. *$T = T_n$ satisfies*

$$\int dx e^{10\gamma_0|x|} \|(\mathbf{S}_\ell T^m)(x)\|_{\mathcal{G}} \leq C, \quad 1 \leq m \leq \ell^2 \quad (7.1)$$

$$\sup_x e^{20\gamma_0|x|} \|(\mathbf{S}_\ell T^{\ell^2})(x)\|_{\mathcal{G}} \leq C \quad (7.2)$$

$$\sup_x e^{\frac{20\gamma_0}{\sqrt{m}}|x|} \|T^m(x)\|_{\mathcal{G}} \leq C m^{-d/2}, \quad 20 \leq \sqrt{m} \leq \ell \quad (7.3)$$

Proof. Let us first suppose $\max(20, \frac{16d}{\alpha}) \leq \sqrt{m} \leq \ell$. By analyticity

$$e^{\frac{20\gamma_0}{\sqrt{m}}|x|} T^m(x) = \int_{\mathbb{T}_n} dp \hat{T}(p + i \frac{20\gamma_0}{\sqrt{m}} e_x)^m e^{ipx} \quad (7.4)$$

where $e_x = x/|x|$. To evaluate this integral let $\chi_n(p)$ be the indicator function for $|\operatorname{Re} p| \leq \mathfrak{p}_n$. On its support (and with $|\operatorname{Im} p| \leq \gamma_0$ we have $\hat{T}(p)^m = e^{mf(p)} R(p) + ((1 - R(p))\hat{T}(p))^m$ and using (6.15)

$$\int_{\mathbb{T}_n} dp \chi_n(p) \|\hat{T}(p + i \frac{20\gamma_0}{\sqrt{m}} e_x)^m\|_{\mathcal{G}} \leq C e^{C\gamma_0^2} \int_{\mathbb{R}^d} dp e^{-\frac{1}{2} D_n m p^2} + (\frac{1}{2} \mathfrak{b}_n)^m \mathfrak{p}_n^d \leq C(m^{-d/2} + \delta^m) \quad (7.5)$$

with $\delta < 1$ (since \mathfrak{p}_n increases sublinearly with n).

To perform the integral with $1 - \chi_n$, we use first (6.13) and $m > 20$, and then $m > \frac{8d}{\alpha}$, to obtain

$$\int_{\mathbb{T}_n} dp (1 - \chi_n(p)) \|\hat{T}^m(p + i \frac{20\gamma_0}{\sqrt{m}} e_x)\|_{\mathcal{G}} \leq \mathfrak{b}_n^m \operatorname{Vol}(\mathbb{T}_n) \leq C^m \ell^{-\frac{\alpha}{8} m n} \ell^{dn} \leq \mathfrak{b}_n^{m/2} \quad (7.6)$$

where $\text{Vol}(\mathbb{T}_n) = (2\pi)^d \ell^{dn}$ is the volume of the d -dimensional torus $\mathbb{T}_n = \ell^n \mathbb{T}^d$. Combining these bounds we get a bound $O(m^{-d/2})$ for (7.4) which implies (7.3) for $\max(20, \frac{16d}{\alpha}) \leq \sqrt{m}$. Taking $m = \ell^2$ and using that $(\mathcal{S}_\ell T^m)(x) = \ell^d T^m(\ell x)$, we get (7.2).

To get (7.1), we first note that, from (7.3) we get immediately

$$\int dx e^{\frac{10\gamma_0}{\sqrt{m}}|x|} \|T^m(x)\|_{\mathcal{G}} \leq C \quad (7.7)$$

After a change of variables $x = \ell y$ and using $\frac{10\ell\gamma_0}{\sqrt{m}} \geq 10\gamma_0$ (7.1) follows for $\sqrt{m} \geq \max(20, \frac{16d}{\alpha})$. For $20 \leq \sqrt{m} \leq \frac{16d}{\alpha}$ we get $\|T^m(x)\|_{\mathcal{G}} e^{\frac{1}{2}\gamma_0|x|} \leq C$ from eq. (6.14). This settles (7.3) and, upon scaling, also (7.1). \square

For later purposes we register the following consequence of (7.5) and (7.6):

$$\|(\mathcal{S}_\ell T_n^{\ell^2})(x) - \tilde{T}_{n+1}(x)\|_{\mathcal{G}} \leq C \ell^{-cn} e^{-20\gamma_0|x|} \quad (7.8)$$

where $(\tilde{p} := p + 20i\gamma_0 e_x)$

$$\tilde{T}_{n+1}(x) = e^{-20\gamma_0|x|} \int_{\mathbb{T}_{n+1}} dp R_n(\tilde{p}/\ell) e^{\ell^2 f_n(\frac{\tilde{p}}{\ell})} e^{ipx} \chi_n(p/\ell). \quad (7.9)$$

7.2 Contribution to T_{n+1} from G_A^c

We recall the expression for T_{n+1} in terms of quantities at scale n ;

$$T_{n+1} = \mathcal{S}_\ell T_n^{\ell^2} + \sum_{A \in \mathfrak{B}(I_{\tau'})} \mathcal{T}_{\tau'} \mathcal{S}_\ell \left[\bigotimes_{A \in \mathcal{A}} G_{n,A}^c \bigotimes_{\tau \notin \text{Supp} \mathcal{A}} T_n(\tau) \right] \quad (7.10)$$

and below we define E_n as the second term on the RHS.

We now derive the necessary bounds on E . This bound is the only place in the induction step $T \rightarrow T'$ in which we need information on the cumulants G_A^c .

Lemma 7.2. *Let*

$$E_n := T_{n+1} - \mathcal{S}_\ell [T_n^{\ell^2}]. \quad (7.11)$$

Then

$$\sup_x e^{10\gamma_0|x|} \|E_n(x)\|_{\mathcal{G}} \leq C \epsilon_n \quad (7.12)$$

and hence in particular $\|E_n\|_{\gamma_0} \leq C \epsilon_n$.

Proof. We (again) abbreviate $T = T_n$, $E = E_n$ and $G_A^c = G_{n+1,A}^c$. We take $\ell > 10$ and apply Lemma 5.3 to eq. (3.34):

$$\left\| \mathcal{T}_{\tau'} \left[\bigotimes_{A \in \mathcal{A}} G_A^c \bigotimes_{\tau \notin \text{Supp} \mathcal{A}} T(\tau) \right] \right\|_{10\gamma_0} \leq \prod_{A \in \mathcal{A}} \|G_A^c\|_{\gamma_0} \prod_J \|T^{|J|}\|_{\gamma_0} \quad (7.13)$$

where the product \prod_J runs over all discrete intervals J in the sets $I_{\tau'} \setminus \text{Supp} \mathcal{A}$. That is; decompose the set $I_{\tau'} \setminus \text{Supp} \mathcal{A}$ into a union of sets J of consecutive numbers (“discrete intervals”) such that no two of them are consecutive. By invoking Lemma 7.1, we bound

$$\|T^{|J|}\|_{\gamma_0} \leq C, \quad \text{since } |J| \leq \ell^2 \quad (7.14)$$

The number of stretches is at most $1 + |\text{Supp} \mathcal{A}| < 2|\text{Supp} \mathcal{A}|$ and hence we can bound

$$\|E\|_{10\gamma_0} \leq \sum_{A \in \mathfrak{B}(I_{\tau'})} C^{|\text{Supp} \mathcal{A}|} \prod_{A \in \mathcal{A}} \|G_A^c\|_{10\gamma_0} \leq C \frac{\ell^2 \epsilon_n^2}{1 - C \ell^2 \epsilon_n^2} \quad (7.15)$$

where the second inequality follows from Proposition 6.2 and the fact that $\sum_{A \in \mathcal{A}} |A|$ is at least 2.

To get a bound on $\sup_x \|E(x)\|_{\mathcal{G}} e^{\gamma|x|}$, we must proceed differently; we split the contributions to $E = E_I + E_{II}$ in (7.10) into those where there is at least one T , namely when $\text{Supp}\mathcal{A} \neq I_{\tau'}$, and those for which there are no T 's, i.e., $\text{Supp}\mathcal{A} = I_{\tau'}$. For the second term, E_{II} , we use that

$$\|E_{II}\|_{10\gamma_0} \leq \sum_{\mathcal{A} \in \mathfrak{B}(I_{\tau'}), \text{Supp}\mathcal{A} = I_{\tau'}} C^{|\text{Supp}\mathcal{A}|} \prod_{A \in \mathcal{A}} \|G_A^c\|_{10\gamma_0} \leq C^{\ell^2} \ell^{2\ell^2} \epsilon_n^{\ell^2} \quad (7.16)$$

where $\ell^{2\ell^2}$ originates as a crude bound for the number of partitions of ℓ^2 elements (bounding the sum over $\mathcal{A} \in \mathfrak{B}(I_{\tau'})$). Further, remark that E_{II} is translation invariant and use the a priori bound on the supremum norms given by

$$\sup_x \|E_{II}(x)\|_{\mathcal{G}} e^{\gamma|x|} \leq \ell^{dn} \int_{\mathbb{X}_n} dx \|E_{II}(x)\|_{\mathcal{G}} e^{\gamma|x|} = \ell^{dn} \|E_{II}\|_{\gamma} \quad (7.17)$$

This bound originates from the fact that the 'unit cell volume' of the lattice \mathbb{X}_n is ℓ^{-dn} . Using now the bound on $\|E_{II}\|_{10\gamma_0}$ derived above, we get

$$\sup_{x, x'} |E_{II}(x' - x)| e^{10\gamma_0|x-x'|} \leq \ell^{dn} (C\ell^2\epsilon_n)^{\ell^2} \leq C\epsilon_0 \ell^{-cn\ell^2\tilde{\alpha}} \leq \epsilon_n \quad (7.18)$$

where the second and third inequality follows from $\epsilon_n = \epsilon_0 \ell^{-\tilde{\alpha}n}$ (by Proposition 6.2).

We now deal with E_I . Here one uses the bound

$$\sup_x \|E_I(x' - x)\|_{\mathcal{G}} e^{10\gamma_0|x-x'|} \leq \sum_{\mathcal{A} \in \mathfrak{B}(I_{\tau'}), \text{Supp}\mathcal{A} \neq I_{\tau'}} \prod_{A \in \mathcal{A}} \|G_A^c\|_{\gamma_0} \left(\sup_{x, x'} \|T(x' - x)\|_{\mathcal{G}} e^{\gamma_0|x-x'|} \right) \quad (7.19)$$

$$\leq C\epsilon_n \sup_x \|T(x)\|_{\mathcal{G}} e^{\gamma_0|x|} \leq C\epsilon_n \quad (7.20)$$

The second inequality is obtained by summing \mathcal{A} as in (7.15). To get the first inequality, first apply point 2) of Lemma 5.2 in all s -coordinates and then apply point 3) of the same Lemma in the x -coordinates. \square

7.3 Analysis of T_{n+1}

We complete the induction step for T by establishing

Lemma 7.3. *Proposition 6.1 holds on scale $n+1$.*

Proof. Whenever confusion is excluded, we abbreviate $T = T_n, R = R_n$. From Lemma 7.2 we get

$$\hat{T}_{n+1}(p) = \hat{T}_n(p/\ell)^{\ell^2} + \hat{E}_n(p), \quad \text{with } \sup_{\text{Im } p \leq 2\gamma_0} \|\hat{E}_n(p)\|_{\mathcal{G}} \leq C\epsilon_n. \quad (7.21)$$

Let first $|\text{Re } p| \geq \mathfrak{p}_{n+1}$. To study the first term in (7.21) with Proposition 6.1 we need to consider two cases:

1. $|\text{Re } p| \geq \ell \mathfrak{p}_n$. By (6.13)

$$\|\hat{T}(p/\ell)^{\ell^2}\|_{\mathcal{G}} \leq \mathfrak{b}_n^{\ell^2} < \frac{1}{2} \mathfrak{b}_{n+1}. \quad (7.22)$$

Now use $C\epsilon_n = C\epsilon_0 \ell^{-\tilde{\alpha}n} \leq \frac{1}{2} \mathfrak{b}_{n+1}$ by taking $\epsilon_0 \leq c(\ell)$ to get (6.13) for $n+1$.

2. $\mathfrak{p}_{n+1} \leq |\text{Re } p| \leq \ell \mathfrak{p}_n$. By Proposition 6.1 2) we have in this region

$$\hat{T}(p/\ell)^{\ell^2} = R(p/\ell) e^{\ell^2 f_n(\frac{p}{\ell})} + (1 - R(p/\ell)) \hat{T}(p/\ell)^{\ell^2} \quad (7.23)$$

To control the first term on the RHS, we use (6.10), (6.13) and (6.15) to get

$$\|R(p/\ell) e^{\ell^2 f_n(p/\ell)}\|_{\mathcal{G}} \leq e^{-\frac{1}{2} D_n \mathfrak{p}_{n+1}^2} e^{C\gamma_0^2} \leq C\ell^{-\frac{1}{4}\tilde{\alpha}(n+1)} \leq \frac{1}{3} \mathfrak{b}_{n+1}. \quad (7.24)$$

The second term on the RHS of (7.23) is bounded by

$$\|(1 - R(p/\ell))\hat{T}(p/\ell)^{\ell^2}\|_{\mathcal{G}} \leq \mathfrak{b}_n^{\ell^2} \leq \frac{1}{3}\mathfrak{b}_{n+1} \quad (7.25)$$

Again, we may assume $C\epsilon_n \leq \frac{1}{3}\mathfrak{b}_{n+1}$ and so (6.13) holds for $n+1$.

Let then $|\operatorname{Re} p| \leq \mathfrak{p}_{n+1}$. We apply spectral perturbation theory, see Lemma A.1. Referring to the notation in that lemma, we have $A = \hat{T}_{n+1}(p)$ and $A_0 = \hat{T}_n(p/\ell)^{\ell^2}$, $a_0 = e^{\ell^2 f_n(p/\ell)}$, $P_0 = R_n(p/\ell)$ and $A_1 = E_n(p)$.

Let us first consider p in a wider strip $|\operatorname{Re} p| \leq \mathfrak{p}_{n+1}$ and $|\operatorname{Im} p| \leq 2\gamma_0$. In this region by (6.15),

$$|a_0| = |e^{\ell^2 f_n(p/\ell)}| \geq e^{-\frac{3}{2}D_n \mathfrak{p}_{n+1}^2 - C\gamma_0^2} > c\ell^{-\frac{3\tilde{\alpha}}{4}(n+1)} \quad (7.26)$$

whereas from (6.7)

$$\|A_0 - a_0 P_0\|_{\mathcal{G}} \leq \mathfrak{b}_n^{\ell^2} = \left(\frac{1}{2}\right)^{\ell^2} \ell^{-\ell^2 \frac{\tilde{\alpha}}{8} n}. \quad (7.27)$$

This is no bigger than $|a_0|/2$ say and so (A1), the first condition of the Lemma A.1, holds. Because of $\|A_1\| \leq C\epsilon_n$ and (6.6), the condition (A13) holds if

$$C\epsilon_n \leq |a_0|$$

which is true for small ϵ_0 . Lemma A.1 then implies that the isolated eigenvalue persists and

$$\left| e^{f_{n+1}(p)} - e^{\ell^2 f_n(\frac{p}{\ell})} \right| \leq C\epsilon_n, \quad (7.28)$$

$$\|R_{n+1}(p) - R_n(p/\ell)\|_{\mathcal{G}} \leq C\epsilon_n(|a_0| - \|A_0 - a_0 P_0\|_{\mathcal{G}} - C\epsilon_n)^{-1} \leq C\epsilon_0 \ell^{-n\tilde{\alpha}} \ell^{\frac{3}{4}\tilde{\alpha}(n+1)} \quad (7.29)$$

so the first claim in (6.9) follows. Furthermore,

$$\|(1 - R_{n+1}(p))\hat{T}_{n+1}(p)\|_{\mathcal{G}} \leq C\|R_{n+1}(p) - R_n(p/\ell)\|_{\mathcal{G}} + C\mathfrak{b}_n^{\ell^2} + C\epsilon_n \leq \mathfrak{b}_{n+1}. \quad (7.30)$$

It remains to iterate (6.8). The analyticity of f_{n+1} follows immediately. To derive the symmetry properties of f_{n+1} , consider a lattice symmetry O . We start from the symmetry property (cfr. Section 5.3)

$$T_{n+1} = \operatorname{Ad}(V_O^{-1})T_{n+1}\operatorname{Ad}(V_O) \quad (7.31)$$

By the result in Section 5.3.3, it follows then that

$$\hat{T}_{n+1}(O^{-1}p) = I_{p,O}^{-1} \hat{T}_{n+1}(p) I_{p,O} \quad (7.32)$$

which implies that $f_{n+1}(p) = f_{n+1}(O^{-1}p)$, and hence the Taylor expansion of $f_{n+1}(p)$ around $p = 0$ contains no odd powers.

Furthermore, by (7.26) and (7.28), we have $f_{n+1}(p) = \ell^2 f_n(p/\ell) + j(p)$ with

$$\sup_{|\operatorname{Im} p| \leq 2\gamma_0} |j(p)| \leq C\epsilon_n \ell^{\frac{3}{4}\tilde{\alpha}(n+1)} \leq \sqrt{\epsilon_0} \ell^{-\frac{1}{4}\tilde{\alpha}n} \quad (7.33)$$

The diffusion constant D_{n+1} is defined as the quadratic term in the Taylor expansion: $D_{n+1}\delta_{i,j} = \frac{1}{2}\nabla_i \nabla_j f_{n+1}(p=0)$. By Cauchy,

$$|D_{n+1} - D_n| \leq 2\gamma_0^{-2} \sup_{|p| \leq 2\gamma_0} |j(p)| \quad (7.34)$$

and the second claim of (6.9) follows by the bound (7.33). Furthermore, for $|\operatorname{Im} p| \leq \gamma_0$ we have

$$|f_{n+1}(p) + D_{n+1}p^2| \leq |\ell^2 f_n(p/\ell) + D_n p^2| + |j(p) - \frac{1}{2} \sum_{i,j} p_i p_j (\nabla_i \nabla_j j)(0)| \quad (7.35)$$

$$\leq (\ell^{-1} \ell^{-\frac{\tilde{\alpha}}{4}n} + 6\gamma_0^{-3} \sup_{|\operatorname{Im} p| \leq 2\gamma_0} |j(q)|) |p|^3 \quad (7.36)$$

which proves (6.8) at $n + 1$ because of (7.33).

Eq. (6.11) follows since T_n necessarily preserves the trace of density matrices and $\text{Tr } \rho = \sum_{s \in \mathcal{S}_0} \hat{\rho}(p = 0, s)$. The bound (6.12) iterates due to (7.29) and will be established for $n = 0$ in Section 11.3.

We complete the proof of items 2) and 3) of Proposition 6.1 by noting that the constants C in the bounds (6.6, 6.12) may be chosen uniform in n by invoking (6.9) for all $n' \leq n$. To get item 1) of Proposition 6.1, note similarly that the proofs of 2) and 3) give for all p with $|\text{Im } p| \leq \gamma_0$

$$\|\hat{T}_{n+1}(p) - \hat{T}_n(p)\|_{\mathcal{G}} \leq C\ell^{-cn}$$

with $c > 0$. Since this holds with n replaced by n' for all $n' \leq n$, $\hat{T}_{n+1}(p)$ can be bounded by an n -independent constant. The bound on $\hat{T}_{n+1}(0, p)$ propagates trivially due to $\hat{T}_{n+1}(0, p) = \hat{T}_n(0, p/\ell)$.

It then remains to prove item 4). For this we use the L^∞ -bound on $E(x)$ (Lemma 7.2), and the result (7.8). Together they show

$$e^{10\gamma_0|x|} \|T_{n+1}(x) - \tilde{T}_{n+1}(x)\|_{\mathcal{G}} \leq C\ell^{-cn}.$$

Consider now $\tilde{T}_{n+1}(x)$ defined in (7.9). From (6.8) and (7.33) we get $|\text{Re } \ell^2 f_n(\tilde{p}/\ell) + D_n p^2| \leq C$ on the support of χ_n . Hence

$$e^{\gamma_0|x|} \|\tilde{T}_{n+1}(x)\|_{\mathcal{G}} \leq CD_n^{-d/2} \sup_{|\text{Re } p| \leq p_n, |\text{Im } p| \leq \gamma_0} \|R_n(p)\|_{\mathcal{G}}.$$

These estimates show 4) holds uniformly in n .

This finishes the proof of the induction step $n \rightarrow n + 1$ for Proposition 6.1 on the condition of the bounds of Proposition 6.2 at level n . \square

8 Flow of G_A^c : Linear part

We will now analyze the recursion relation for the correlation functions G_A^c given in (3.34). In the present section and in Section 9, we drop the subscript n on operators, writing e.g. $G_A^c = G_{n,A}^c, T = T_n$ and we denote operators on scale $n + 1$ by a prime, e.g. $G_{A'}^{c'} = G_{n+1,A}^c$.

We split the recursion relation into a *linear* and a *nonlinear* part (although these adjectives do not fit literally). Consider the terms in the sum in (3.34) with \mathcal{A} a collection that consists of just one set, namely $\mathcal{A} = \{A\}$. We use the notation $A \rightarrow A'$ to indicate that A contributes to $G_{A'}^{c'}$ in this sense. In other words,

$$A \rightarrow A' \quad \Leftrightarrow \quad A \in I_{A'} \text{ and } \forall \tau' \in A' : (A \cap I_{\tau'}) \neq \emptyset. \quad (8.1)$$

The *linear* RG flow is the contribution of such \mathcal{A} , with the additional restriction that $|A'| = |A|$:

$$G_{A',\text{lin}}^{c'} := \sum_{A: A \rightarrow A', |A'| = |A|} \mathcal{S}_\ell \mathcal{T}_{A'} \left[G_A^c \otimes_{\tau \in I_{A'} \setminus A} T(\tau) \right] \quad (8.2)$$

The aim of this section is to state good bounds on $\|G_{A',\text{lin}}^{c'}\|_{10\gamma_0}$, such that the induction hypothesis Proposition 6.2 can be carried from scale to scale if G_A^c is replaced by $G_{A,\text{lin}}^c$. We call the remaining terms in the sum (3.34) nonlinear and we treat them in Section 9. By inspecting Proposition 6.2, it is easy to understand the qualitative difference between the linear and nonlinear contributions. Recall that in Proposition 6.2 we claim a bound proportional to $\epsilon_n^{|A|}$. The nonlinear contributions to $G_{A'}^{c'}$ carry at least $|A'| + 1$ powers of ϵ_n and this gives us some leeway. However, the linear contribution has just $|A'|$ factors of ϵ_n and this forces us to do a careful analysis to be able to get a bound with $\epsilon_{n+1}^{|A'|}$. There are two mechanisms at work that help us here:

- *Rescaling of time:* Since the microscopic times described by the macrotimes τ on scale $n + 1$ are ℓ^2 larger than those on scale n , the blow-up factor $\text{dist}(A)^\alpha$ for $A \rightarrow A'$ is roughly speaking $\ell^{2\alpha(|A'| - 1)}$ times larger than $\text{dist}(A')^\alpha$.

- *Ward Identities:* The Ward identities from unitarity and (in the case $\beta_1 = \beta_2$) reversibility allow to improve our estimate on each term in the sum on the RHS of (8.2). The gain from each of the Ward identities is roughly ℓ^{-1} . (this will be explained later)

The main force working against us is

- *Entropy factors:* For each A' , there are $\ell^{2|A'|}$ sets A such that $A \rightarrow A'$, $|A| = |A'|$. However, due to the summability over A with $\min A$ fixed in Proposition 6.2, only one factor ℓ^2 (corresponding to $\min A$), shows up in our estimates.

The gain from the Ward Identities is explained in Section 8.1. Then, in Sections 8.2 and 8.3, we prove Propositions 8.1, 8.2, 8.3, which we present now.

Proposition 8.1 provides a bound on $G_{A', \text{lin}}^{c'}$ which is valid regardless of whether ρ_E^{ref} is an equilibrium state or not. For this reason we call this bound 'non-equilibrium'. In what follows, we denote by $\lfloor x \rfloor$ the largest integer smaller than x .

Proposition 8.1 (Non-equilibrium linear RG).

$$\sum_{A' \subset \mathbb{N}; |A'|=k, \min A'=1} \text{dist}(A')^\alpha \|G_{A', \text{lin}}^{c'}\|_{10\gamma_0} \leq C^k \epsilon_n^k \ell^{1-4\alpha} \ell^{-2\alpha \lfloor \frac{k-3}{2} \rfloor} (\log^C \ell), \quad k > 2 \quad (8.3)$$

$$\sum_{A' \subset \mathbb{N}; |A'|=2, \min A'=1} \text{dist}(A')^\alpha \|G_{A', \text{lin}}^{c'}\|_{10\gamma_0} \leq C \ell^{1-2\alpha} \epsilon_n^2 (\log^C \ell) \quad (8.4)$$

Note that the bound in the case $|A'| = 2$ is only useful when $\alpha > 1/2$, since for $\alpha \leq 1/2$ the power of ℓ on the RHS becomes nonnegative. The next two propositions are only meaningful in the equilibrium case $\beta_1 = \beta_2 = \beta$. Proposition 8.2 deals with correlation functions $G_{n,A}^c$ with $A \ni 0$.

Proposition 8.2 (Initial time linear RG). *Assume $\beta_1 = \beta_2 = \beta$, then*

$$\sum_{A' \subset \mathbb{N}_0; |A'|=k, \min A'=0} \text{dist}(A')^\alpha \|G_{A', \text{lin}}^{c'}\|_{10\gamma_0} \leq C^k \epsilon_n^k \epsilon_{1,n} \ell^{-2\alpha \lfloor \frac{k-1}{2} \rfloor} (\log^C \ell), \quad k > 2 \quad (8.5)$$

$$\sum_{A' \subset \mathbb{N}_0; |A'|=2, \min A'=0} \text{dist}(A')^\alpha \|G_{A', \text{lin}}^{c'}\|_{10\gamma_0} \leq C \epsilon_n^2 \epsilon_{1,n} \ell^{-\min(1, 2\alpha)} (\log^C \ell) \quad (8.6)$$

The improved bound in Proposition 8.2 (compared to Proposition 8.1) is due to the fact that the macroscopic time 0 does not get rescaled, and hence there is essentially no entropy factor from sets A containing 0.

Finally, we state a bound that applies to bulk sets A , but that is better than Proposition 8.1 since it allows to treat the $|A'| = 2$ correlation function for $\alpha < 1/2$.

Proposition 8.3 (Equilibrium linear RG). *Assume $\beta_1 = \beta_2 = \beta$ and that $0 < \alpha \leq 1/2$. Then*

$$\sum_{A' \subset \mathbb{N}; |A'|=2, \min A'=1} \text{dist}(A')^\alpha \|G_{A', \text{lin}}^{c'}\|_{10\gamma_0} \leq C \epsilon_n^2 \ell^{-2\alpha} (\log^C \ell) \quad (8.7)$$

The improved bound in Proposition 8.3 is due to the fact that we can use two Ward identities instead of one.

If one neglects the contribution of the nonlinear part in the recursion relation (3.34), then the above bounds establish the induction hypothesis Proposition 6.2. Indeed, for $1/2 < \alpha < 1$ (the case $\beta_1 \neq \beta_2$), Proposition 8.1 implies the bound (6.17) of Proposition 6.2 provided that

$$1 - 2\alpha < -2\tilde{\alpha} \quad (8.8)$$

$$(1 - 4\alpha) - 2\alpha \lfloor (k-3)/2 \rfloor < -k\tilde{\alpha}, \quad k > 2 \quad (8.9)$$

and this is indeed satisfied for our choice $\tilde{\alpha} = \alpha/2 - 1/4$. For $1/4 < \alpha \leq 1/2$ (the case $\beta_1 = \beta_2$), we need (combining Propositions 8.1 and 8.3)

$$-2\alpha < -2\tilde{\alpha} \quad (8.10)$$

$$(1 - 4\alpha) - 2\alpha \lfloor (k-3)/2 \rfloor < -k\tilde{\alpha}, \quad k > 2 \quad (8.11)$$

to satisfy the bound (6.17) and (by Proposition 8.2)

$$-2\alpha < -2\tilde{\alpha} - \tilde{\alpha}_I \quad (8.12)$$

$$-2\alpha \lfloor (k-1)/2 \rfloor < -k\tilde{\alpha} - \tilde{\alpha}_I, \quad k > 2 \quad (8.13)$$

to satisfy the bound (6.18). One inspects that $\tilde{\alpha} = \alpha/4 - 1/8$ and $\tilde{\alpha}_I = 1/4$ do the job.

8.1 Contraction from the Ward Identities

Consider the connected correlation function $G_A^c \in \mathcal{R}_A$ (we have dropped the subscript n again). As explained in Section 4.1, it satisfies the Ward identity

$$\int dx'_\tau (P \otimes 1 \dots \otimes 1) G_A^c(x'_A, x_A) = 0, \quad \tau = \max A \quad (8.14)$$

where P is any projector in \mathcal{G} of the form $P = |\mu\rangle\langle 1_{S_0}|$ with $\sum_{s \in S_0} \mu(s) = 1$, as introduced in Section 4.1. We will use this identity to derive a bound on the operator

$$(T^m \otimes 1 \otimes \dots \otimes 1) G_A^c \quad (8.15)$$

where T^m acts on the leg of the tensor product indexed by $\tau = \max A$. The relevance of (8.15) is that it comes up after writing out the RHS of (8.2) explicitly, i.e. performing the contraction $\mathcal{T}_{A'}$. However, for the purpose of the present section, one can just accept (8.15) as the basic object of study. The reason why one can get a bound on (8.15) is that, formally speaking, T^m has a right eigenvector with eigenvalue 1 given by $1_{S_0} = 1_{[s \in S_0]}$ (in particular, it is independent of x) and it contracts vectors orthogonal to 1_{S_0} to size $\sim m^{-1/2}$. The condition (8.14) ensures that $G_{n,A}^c$ as a function of x'_τ is orthogonal to 1_{S_0} and hence (8.15) should decay $\sim m^{-1/2}$. This intuition is captured by

Lemma 8.1. *Let $K \in \mathcal{R}$ be a kernel satisfying the 'Ward Identity'*

$$\int dx' PK(x', x) = 0, \quad (8.16)$$

Then, there is an operator $V_{m,\gamma} \in \mathcal{R}$ satisfying

$$\|\mathbf{S}_\ell V_{m,\gamma_0/2}\|_{10\gamma_0} \leq \frac{C \log(1+m)}{\sqrt{m}}, \quad \text{for } 1 \leq m \leq \ell^2. \quad (8.17)$$

and such that

$$|T^m K| \leq |V_{m,\gamma}| |K e^{\gamma|x'-x|}|, \quad (8.18)$$

where we recall that $|L|$ is the operator whose kernel is the absolute value of the kernel of L , and the inequality in the last formula is between kernels.

The proof of this lemma is postponed to Section 8.1.1.

Now it is straightforward to bound (8.15) (after rescaling), indeed, let $\gamma = 10\gamma_0$, then

$$\|\mathbf{S}_\ell (T^m \otimes 1 \otimes \dots \otimes 1) G_A^c\|_{10\gamma_0} = \| |(T^m \otimes 1 \otimes \dots \otimes 1) G_A^c| \|_{10\gamma_0/\ell} \quad (8.19)$$

$$\leq (\|V_{m,\gamma_0/2} \otimes 1 \otimes \dots \otimes 1\| (G_A^c e^{\frac{\gamma_0}{2}|x'_{\max A} - x_{\max A}|})) \|_{10\gamma_0/\ell} \quad (8.20)$$

$$\leq \frac{C \log^C(1+m)}{\sqrt{m}} \|G_A^c\|_{\frac{10\gamma_0}{\ell} + \frac{\gamma_0}{2}} \quad (8.21)$$

The first inequality uses (8.18) with the coordinates $z_{A \setminus \max A}, z'_{A \setminus \max A}$ kept fixed. The second inequality uses (8.17) and the definition of the norm $\|\cdot\|_\gamma$.

Let us now try an analogous trick based on the Ward identity (4.31) which we rewrite here with the help of the projectors $P^\beta := |\mu^\beta\rangle\langle 1_{S_0}|$ and $P^{\text{ref}} = |\mu^{\text{ref}}\rangle\langle 1_{S_0}|$;

$$\int dx_a G_{\{1,\tau\}}^c(x_a', x_b'; x_a, x_b)(1 \otimes P^\beta) = \int dx_a L_\tau(x_a', x_b'; x_a, x_b)(1 \otimes P^{\text{ref}}) \quad (8.22)$$

It will be used to get smallness for the right action by T^m i.e. for $G_{\{1,\tau\}}^c(1 \otimes T^m)$ by a similar reasoning to the one given above for the Ward identity from unitarity. The operator T formally has a unique left eigenvector with eigenvalue 1, given by μ_{T_n} as defined in 6.11, and T^m contracts vectors orthogonal to μ_{T_n} to size $\sim m^{-1/2}$. Hence, if (8.22) would hold with the RHS replaced by 0 and P^β replaced by $R(0)$, then we could repeat the reasoning of the unitarity Ward identity. However, the projection $P^\beta = |\mu^\beta\rangle\langle 1_{S_0}|$ is $\sqrt{\epsilon_0}$ -close to $R(0) = |\mu_{T_n}\rangle\langle 1_{S_0}|$, see (6.12) in the induction hypothesis. Actually μ_{T_n} converges to μ^β as $n \rightarrow \infty$, but this is not even needed here. The presence of the RHS in (8.22) introduces an additional error term which is small because L_τ consists of correlation functions that include $\tau = 0$ and therefore contract faster.

Lemma 8.2. *Let $K, \tilde{K} \in \mathcal{R}$ be kernels satisfying the 'Ward Identity'*

$$\int dx K(x', x) P^\beta = \int dx \tilde{K}(x', x) P^{\text{ref}} \quad (8.23)$$

Then, there are operators $W_{m,\gamma}, \tilde{W}_{m,\gamma} \in \mathcal{R}$ satisfying

$$\|\mathbf{S}_\ell W_{m,\gamma_0/2}\|_{10\gamma_0} \leq Cm^{-1/2} \log m + C\|R(0) - P^\beta\|_{\mathcal{G}}, \quad (8.24)$$

$$\|\mathbf{S}_\ell \tilde{W}_{m,\gamma_0/2}\|_{10\gamma_0} \leq C \quad (8.25)$$

for $1 \leq m \leq \ell^2$, and

$$|KT^m| \leq |Ke^{\gamma|x-x'|}|W_{m,\gamma} + |\tilde{K}e^{\gamma|x-x'|}|\tilde{W}_{m,\gamma} \quad (8.26)$$

We will use the Lemma to derive a bound on the correlation function $(T^{m+} \otimes 1)G_{\{1,\tau\}}^c(1 \otimes T^{m-})$. Its kernel is bounded

$$|(T^{m+} \otimes 1)G_{\{1,\tau\}}^c(1 \otimes T^{m-})| \leq V_{m+,\gamma} |e^{\gamma|x'_\tau - x_\tau|} G_{\{1,\tau\}}^c(1 \otimes T^{m-})| \quad (8.27)$$

$$\leq V_{m+,\gamma} \left(|e^{\gamma(|x'_\tau - x_\tau| + |x'_1 - x_1|)} G_{\{1,\tau\}}^c|W_{m-,\gamma} + |e^{\gamma|x'_1 - x_1|} L_\tau| \tilde{W}_{m-,\gamma} \right) \quad (8.28)$$

where we used first (8.18) and then (8.26). Collecting the bounds on V, W, \tilde{W} , we get

$$\begin{aligned} & \|\mathbf{S}_\ell (T^{m+} \otimes 1) G_{\{1,\tau\}}^c(1 \otimes T^{m-})\|_{10\gamma_0} \\ & \leq C \frac{\log m_+}{\sqrt{m_+}} \| |e^{\frac{\gamma_0}{2}|x'_\tau - x_\tau|} G_{\{1,\tau\}}^c(1 \otimes T^{m-}) \|_{10\gamma_0/\ell} \\ & \leq C \frac{\log m_+}{\sqrt{m_+}} \left(\|G_{\{1,\tau\}}^c\|_{\frac{10\gamma_0}{\ell} + \frac{\gamma_0}{2}} \left(\frac{\log m_-}{\sqrt{m_-}} + \|R(0) - P^\beta\|_{\mathcal{G}} \right) + \|L\|_{\frac{10\gamma_0}{\ell}} \right) \end{aligned} \quad (8.29)$$

The operator norm $\|L_\tau\|_\gamma$ can be bounded by norms of correlation functions:

$$\|L_\tau\|_\gamma \leq \|G_{\{0,1,\tau\}}^c\|_\gamma + \|G_{\{0,\tau\}}^c\|_\gamma + \|G_{\{0,\tau-1\}}^c\|_\gamma \quad (8.30)$$

by (4.31) and Lemma 5.3 (1). The upshot of the calculation in (8.29) is that all terms between the large brackets on the last line are smaller than $\|G_{\{1,\tau\}}^c\|_{\frac{10\gamma_0}{\ell}}$ (which would be the resulting bound if we had used only the unitarity Ward identity). As anticipated before Lemma 8.2, the smallness comes either from the factor $\frac{\log m_-}{\sqrt{m_-}}$, the difference $\|R(0) - P^\beta\|_{\mathcal{G}}$ and the correlation functions contributing to L_τ that are small because they involve the initial time 0 and therefore contract faster, see Proposition 8.2.

8.1.1 Proof of Lemma 8.1

We start from (8.16), i.e. $\int dx PK(x, x_0) = 0$ for any x_0 . This implies trivially that

$$T^m K(x', x_0) = \int dx e^{-\gamma|x-x_0|} (T^m(x', x) - T^m(x', x_0)P) K(x, x_0) e^{\gamma|x-x_0|} \quad (8.31)$$

We now choose the projector P to equal $R(0)$ (the spectral projector of $\hat{T}(p=0)$) and we define $V_{m,\gamma} \in \mathcal{R}$

$$V_{m,\gamma}(x', x) := \sup_{x_0} \left| e^{-\gamma|x-x_0|} (T^m(x-x') - T^m(x_0-x')R(0)) \right| \quad (8.32)$$

where the absolute value $|\cdot|$ is applied to a kernel in s, s' , that is, for an operator D in \mathcal{G} we set $|D|$ to be the operator with kernel $|D|(s', s) := |D(s', s)|$. Note that $V_{m,\gamma}$ is translation invariant by the translation invariance of T^m . From (8.31) and (8.32), we get the inequality (8.18) of Lemma 8.1

It remains to establish the bound on $V_{m,\gamma}$, i.e. (8.17). It suffices to consider \sqrt{m} large enough only, we take $\sqrt{m} \geq 80$. Since the operator $V_{m,\gamma}$ is translation invariant, we can use Lemma 5.4 and hence it suffices to show that

$$s_m := \sup_x e^{\frac{20\gamma_0}{\sqrt{m}}|x|} \|V_{m,\gamma_0/2}(0, x)\|_{\mathcal{G}} \leq C m^{-1/2(d+1)} \log m. \quad (8.33)$$

Indeed, we then get

$$\|V_{m,\gamma_0/2}\|_{10\gamma_0/\ell} \leq s_m \int dx e^{(\frac{10\gamma_0}{\ell} - \frac{20\gamma_0}{\sqrt{m}})|x|} \leq C m^{-1/2} \log m \quad (8.34)$$

which implies (8.17) by scaling.

To prove (8.33) we split $V_{m,\gamma}(x', x) = V'_{m,\gamma}(x', x) + V''_{m,\gamma}(x', x)$ where the terms are defined by restricting the supremum in (8.32) to $\frac{\gamma_0}{2}|x-x_0| \leq \log m$ and $\frac{\gamma_0}{2}|x-x_0| > \log m$ respectively. We bound

$$\begin{aligned} e^{\frac{20\gamma_0}{\sqrt{m}}|x|} |V''_{m,\gamma_0/2}(0, x)| &\leq e^{\frac{20\gamma_0}{\sqrt{m}}|x|} \sup_{x_0: \gamma_0|x-x_0| > \log m} e^{-\frac{\gamma_0}{2}|x-x_0|} |T^m(x) - T^m(x_0)R(0)| \\ &\leq \frac{1}{m} e^{\frac{20\gamma_0}{\sqrt{m}}|x|} |T^m(x)| + \sup_{x_0} \frac{1}{\sqrt{m}} e^{-\frac{\gamma_0}{4}|x-x_0|} e^{\frac{20\gamma_0}{\sqrt{m}}|x_0|} |T^m(x_0)R(0)| \end{aligned}$$

where we used the triangle inequality $|x| \leq |x_0| + |x-x_0|$ and $\sqrt{m} \geq 80$ in the last step. The desired bound (8.33) (without the log) now follows from (7.3).

Next, we consider $V'_{m,\gamma_0/2}$. The argument for showing (8.33) for $V'_{m,\gamma_0/2}$ is analogous to the proof of Lemma 7.1 and we will use some notation from that proof. We start from the momentum space representation

$$T^m(x', x) - T^m(x', x_0)R(0) = \int_{\mathbb{T}_n} dp e^{ip(x-x')} [\hat{T}^m(p) - \hat{T}^m(p)R(0)e^{-ip(x-x_0)}] \quad (8.35)$$

and proceed as in (7.5) and (7.6) to conclude that up to terms exponentially small in m (and n) it suffices to bound

$$\left\| \int_{\mathbb{T}_n} dp e^{m f_n(\bar{p})} e^{ipx} (R(\bar{p}) - R(0)e^{-i\bar{p}(x-x_0)}) \chi_n(p) \right\|_{\mathcal{G}} \quad (8.36)$$

where $\bar{p} = p + \frac{20i\gamma_0}{\sqrt{m}}e_x$. This is bounded by

$$C \int_{\mathbb{T}_n} dp e^{-\frac{D_p}{2}mp^2} (\|R(\bar{p}) - R(0)\|_{\mathcal{G}} + |1 - e^{-i\bar{p}(x-x_0)}|) \chi_n(p). \quad (8.37)$$

using the bound (6.15). Since

$$|1 - e^{-i\bar{p}(x-x_0)}| \leq C(|p| + \gamma_0/\sqrt{m}) \log m$$

the second term on the RHS of (8.37) contributes $\mathcal{O}(m^{-(d+1)/2} \log m)$. For the first term use analyticity of $R(p)$ in a ball of radius γ_0 at the origin to get

$$\|R(\bar{p}) - R(0)\|_{\mathcal{G}} \leq C(|p| + \gamma_0/\sqrt{m} + 1_{|\operatorname{Re} p| > c\gamma_0})$$

and hence the first term on the RHS of (8.37) contributes $\mathcal{O}(m^{-(d+1)/2})$, as well.

8.1.2 Proof of Lemma 8.2

We start from the Ward Identity

$$\int dx' K(x'', x') P^\beta = \int dx' \tilde{K}(x'', x') P^{\text{ref}} \quad (8.38)$$

Then

$$\int dx' K(x'', x') T^m(x', x) \quad (8.39)$$

$$= \int dx' K(x'', x') e^{\gamma|x''-x'|} (T^m(x', x) - P^\beta T^m(x'', x)) e^{-\gamma|x''-x'|} \quad (8.40)$$

$$+ \int dx' \tilde{K}(x'', x') e^{\gamma|x''-x'|} P^{\text{ref}} T^m(x'', x) e^{-\gamma|x''-x'|} \quad (8.41)$$

We now define the operators $W_{m,\gamma}, \widetilde{W}_{m,\gamma}$ by specifying their reduced kernels

$$W_{m,\gamma}(x', x) := \sup_{x''} |T^m(x', x) - P^\beta T^m(x'', x)| e^{-\gamma|x''-x'|} \quad (8.42)$$

$$\widetilde{W}_{m,\gamma}(x', x) := \sup_{x''} |P^{\text{ref}} T^m(x'', x)| e^{-\gamma|x''-x'|} \quad (8.43)$$

(the absolute values on the RHS are again meant as absolute values of kernels on $\mathbb{A}_n \times \mathbb{A}_n$, see the remark below (8.32)) Note that both operators are translation invariant. By completely analogous reasoning as the one leading to (8.18), we get the inequality (8.26).

Now to the bounds on $W_{m,\gamma}, \widetilde{W}_{m,\gamma}$. If P^β is replaced by $R(0)$ in the definition of $W_{m,\gamma}(x', x)$, we get the first term of the bound by an analogous proof as that of Lemma 8.1. The error term due to $R(0) - P^\beta$ is estimated by $C\|R(0) - P^\beta\|_{\mathcal{G}}$ where the constant C originates from bounding $\mathbf{S}_\ell T^m$ with the help of Lemma 7.1. Similarly, the bound on $\widetilde{W}_{m,\gamma_0/2}$ is immediate from Lemma 7.1.

8.2 Preliminaries for the induction step $G_A^c \rightarrow G_{A'}^{c'}$

In this section, we gather some tools for the proofs of Propositions 8.1, 8.2, 8.3. We state the main bound that we will use to control the sum over correlation functions at the lower scale. We abbreviate $k := |A| = |A'|$ and $m_+ := \max I_{A'} - \max A$;

$$\begin{aligned} \|G_{A', \text{lin}}^{c'}\|_{10\gamma_0} &\leq \sum_{A \rightarrow A', |A|=k} \left\| \mathbf{S}_\ell \mathcal{T}_{A'} \left[G_A^c \otimes_{\tau \in I_{A'} \setminus A} T(\tau) \right] \right\|_{10\gamma_0} \\ &\leq \sum_{A \rightarrow A', |A|=k} \|(T^{m_+} \otimes 1 \otimes \dots \otimes 1) G_A^c\|_{10\gamma_0/\ell} \prod_{j=1}^k \|T^{\tau_j - \tau_{j-1} - 1}\|_{10\gamma_0/\ell}, \\ &\leq \sum_{A \rightarrow A', |A|=k} C^k \frac{\log(1 + m_+)}{\sqrt{1 + m_+}} \|G_A^c\|_{\gamma_0} \end{aligned} \quad (8.44)$$

On the second line, we set the dummy $\tau_0 = 0$ or, if $\tau_1 = 0$, then $\tau_0 = -1$. The bound is obtained by proceeding as in the proof of Lemma 7.2, bounding powers T^m by Lemma 7.1, except the power T^{m_+} , which is bounded by using Lemma 8.1, as explained in the beginning of Section 8.1.

Next, we introduce some useful notation. We let $h(A)$ stand for a function of the ordered times τ_1, \dots, τ_k of A with the property that, for $\min A > 0$,

$$h(A) = h(A + \tau) \quad (\text{time-translation invariance}) \quad (8.45)$$

and satisfying the normalization conditions, uniformly in τ_k, τ_1 , respectively,

$$\sum_{0 < \tau_1 < \tau_2 < \dots < \tau_k : \tau_k \text{ fixed}} h(A) \leq 1, \quad \text{and} \quad \sum_{\tau_1 < \tau_2 < \dots < \tau_k : \tau_1 \text{ fixed}} h(A) \leq 1 \quad (8.46)$$

Note that the first condition follows from the second by time-translation invariance. In particular, we will use that the function of $k - 1$ variables obtained by performing the sum $\sum_{\tau_1 : \tau_1 < \tau_2} h(A)$ or $\sum_{\tau_k : \tau_k > \tau_{k-1}} h(A)$ satisfies the same conditions, and hence we will also call it $h(\cdot)$. This is how $h(\cdot)$ will enter: we write

$$\|G_{A', \text{lin}}^{c'}\|_{10\gamma_0} \leq \sum_{A \rightarrow A'} \epsilon^{|A|} \frac{\log(1 + m_+)}{\sqrt{1 + m_+}} \text{dist}(A)^{-\alpha} h(A). \quad (8.47)$$

by using (8.44). Now our task is to estimate the RHS, which is done by elementary analysis in the next sections (note indeed that all operators have disappeared from the RHS). Below, we consistently use the notation τ_1, τ_2, \dots for the ordered elements of A and τ'_1, τ'_2, \dots for those of A' . The variables τ_1, τ_2, \dots range over the sets $I_{\tau'_1}, I_{\tau'_2}, \dots$

8.2.1 Change of variables

We need to sum (8.47). Hence we will have to convert the factor $\text{dist}(A')^\alpha$ into $\text{dist}(A)^\alpha$. In doing this, we will gain small factors of $\ell^{-2\alpha}$. Indeed, we find the inequality

$$\text{dist}(\tau'_1, \tau'_2)^\alpha \leq \ell^{-2\alpha} \text{dist}(A)^\alpha, \quad A \rightarrow \{\tau'_1, \tau'_2\}, \tau'_2 - \tau'_1 > 1 \quad (8.48)$$

That is, if the macroscopic times are not neighbors, then we gain a small factor. However, even if the macroscopic times are neighbors, then we still get a small factor from every second difference of macroscopic times, i.e.

$$\text{dist}(A')^\alpha \leq C^{|A'|} (\ell^{-2\alpha})^{\lfloor \frac{|A'|-1}{2} \rfloor} \text{dist}(A)^\alpha, \quad \text{for any } A \rightarrow A' \text{ with } |A'| = |A| \geq 3 \quad (8.49)$$

Indeed, for all $1 < j < |A'|$, either $|\tau_j - \tau_{j+1}| \geq \frac{\ell}{2} |\tau'_j - \tau'_{j+1}|$ or $|\tau_j - \tau_{j-1}| \geq \frac{\ell}{2} |\tau'_j - \tau'_{j-1}|$. We will refer to these bounds as 'change of variables'.

8.3 Proof of Proposition 8.1

To prove Proposition 8.1, we need to control the sum

$$\sum_{A' : |A'|=k, \max A' \text{ fixed}} \text{dist}(A')^\alpha \sum_{A \rightarrow A'} \frac{1}{\sqrt{1 + \max I_{\tau'_k} - \tau_k}} \text{dist}(A)^{-\alpha} h(A) \quad (8.50)$$

which emerges⁶ from the LHS of (8.3,8.4) by (8.47). We will do this by using the change of variables and the integrability of $h(\cdot)$, both introduced above. Note that we dropped the factor $\log^C(1 + m_+)$ since it can always be estimated by $C' \log^C \ell$ which suffices for our purposes. We also did not write $\epsilon_n^{|A|}$ since this factor just goes through all estimates.

8.3.1 The case $|A'| = 2$

Let us assume first that $\tau'_1 < \tau'_2 - 1$, then (8.50) reduces to

⁶In fact, Proposition 8.1 demands that we keep $\min A'$ fixed. However, by time-translation invariance, the claim with $\max A'$ fixed is equivalent

$$\sum_{\tau'_1: \tau'_1 < \tau'_2 - 1} (1 + \tau'_2 - \tau'_1)^\alpha \sum_{\tau_1 \in I_{\tau'_1}, \tau_2 \in I_{\tau'_2}} h(\tau_1, \tau_2) (1 + \tau_2 - \tau_1)^{-\alpha} (\max I_{\tau'_2} - \tau_2 + 1)^{-1/2} \quad (8.51)$$

$$\leq C \ell^{-2\alpha} \sum_{\tau_1 < \tau_2 - \ell^2} \sum_{\tau_2 \in I_{\tau'_2}} h(\tau_1, \tau_2) (\max I_{\tau'_2} - \tau_2 + 1)^{-1/2} \quad (8.52)$$

$$\leq C \ell^{-2\alpha} \sum_{\tau_2 \in I_{\tau'_2}} (\max I_{\tau'_2} - \tau_2 + 1)^{-1/2} = C \ell^{-2\alpha} \sum_{x=1}^{\ell^2} x^{-1/2} \leq C \ell^{1-2\alpha} \quad (8.53)$$

To get the first inequality, we used the change of variables formula (8.49), the second inequality follows from the properties of $h(\cdot)$, i.e., from (8.46).

Next, let us treat the case $\tau'_2 = \tau'_1 + 1$, then (8.50) gives (we can bound $\text{dist}(A')^\alpha$ by a constant here)

$$C \sum_{\tau_1 \in I_{\tau'_1}, \tau_2 \in I_{\tau'_2}} h(\tau_1, \tau_2) (1 + \tau_2 - \tau_1)^{-\alpha} (\max I_{\tau'_2} - \tau_2 + 1)^{-1/2} \quad (8.54)$$

$$\leq C \sum_{\tau_2} (1 + \tau_2 - \min I_{\tau'_2})^{-\alpha} (\max I_{\tau'_2} - \tau_2)^{-1/2} = \sum_{x=1}^{\ell^2} x^{-\alpha} (1 + \ell^2 - x)^{-1/2} \leq C \ell^{1-2\alpha} \quad (8.55)$$

In the first inequality, we used $(1 + \tau_2 - \tau_1)^{-\alpha} \leq (1 + \tau_2 - \min I_{\tau'_2})^{-\alpha}$, and we bounded the sum over τ_1 by (8.46). The last inequality follows easily after splitting the sum in $1 < x \leq \ell^2/2$ and $\ell^2/2 < x \leq \ell^2$. Putting the two contributions together, we get $C \ell^{1-2\alpha}$ as a bound for (8.50) in the case $|A| = |A'| = 2$.

8.3.2 The case $|A'| = 3$

We start with the subcase $\tau'_1 = \tau'_2 - 1 = \tau'_3 - 2$ which turns out to be the most tricky one. First, we dominate

$$\sum_{\tau_{1,2,3} \in I_{\tau'_{1,2,3}}} \text{dist}(\tau_1, \tau_2, \tau_3)^{-\alpha} (1 + \max I_{\tau'_3} - \tau_3)^{-1/2} h(\tau_1, \tau_2, \tau_3) \quad (8.56)$$

$$\leq \sum_{\tau_{2,3} \in I_{\tau'_{2,3}}} \text{dist}(\tau_2, \tau_3)^{-\alpha} (1 + \tau_2 - \min I_{\tau'_2})^{-\alpha} (1 + \max I_{\tau'_3} - \tau_3)^{-1/2} h(\tau_2, \tau_3) \quad (8.57)$$

by $(1 + \tau_2 - \tau_1)^{-\alpha} \leq (1 + \tau_2 - \min I_{\tau'_2})^{-\alpha}$ and the properties of h , i.e. (8.46). Then, we change variables

$$x = \tau_2 - \min I_{\tau'_2}, \quad y = \max I_{\tau'_3} - \tau_3, \quad 0 \leq x, y < \ell^2 \quad (8.58)$$

such that we have to bound

$$\sum_{x,y=0}^{\ell^2-1} F(x, y), \quad F(x, y) = h(x, 2\ell^2 - y - 1) (x+1)^{-\alpha} (y+1)^{-1/2} (2\ell^2 - x - y - 1)^{-\alpha} \quad (8.59)$$

where we used the invariance of $h(\cdot, \cdot)$ under joint translations of its arguments.

To perform these sums, we define the sets

$$\mathcal{C}_n^a := \{(x, y) \mid (1 - \delta_{n,0})2^n \leq x \leq 2^{n+1}, \quad (1 - \delta_{n,0})2^n \leq y \leq 2/3\ell^2\} \quad (8.60)$$

$$\mathcal{C}_n^b := \{(x, y) \mid (1 - \delta_{n,0})2^n \leq y \leq 2^{n+1}, \quad (1 - \delta_{n,0})2^n \leq x \leq 2/3\ell^2\} \quad (8.61)$$

$$\mathcal{C}^c := \{(x, y) \mid 2\ell^2 - x - y \leq 2/3\ell^2\} \quad (8.62)$$

And we will split

$$\sum_{x,y=0}^{\ell^2-1} F(x,y) \leq \sum_{n=0}^{n^*} \sum_{\mathcal{C}_n^a} F(x,y) + \sum_{n=0}^{n^*} \sum_{\mathcal{C}_n^b} F(x,y) + \sum_{\mathcal{C}^c} F(x,y) \quad (8.63)$$

for $n = 0, \dots, n^*$ with n^* the smallest natural number such that $2^{n^*} \geq \ell^2$. To perform the sum in \mathcal{C}_n^a , we bound

$$\sup_{\mathcal{C}_n^a} \left((2\ell^2 - x - y - 1)^{-\alpha} (x+1)^{-\alpha} (y+1)^{-1/2} \right) = C\ell^{-2\alpha} (2^n)^{-1/2-\alpha}, \quad (8.64)$$

we use properties (8.46) to perform the y -sum and we bound the x -sum by 2^n , the 'width' of its domain. This yields

$$\sum_{\mathcal{C}_n^a} F(x,y) \leq C\ell^{-2\alpha} (2^n)^{1-1/2-\alpha} \quad (8.65)$$

The sum in \mathcal{C}_n^b is done analogously, except that now the x -sum is controlled by (8.46) and the y -sum by its width 2^n . On \mathcal{C}^c , we can bound $(x+1)^{-\alpha} (y+1)^{-1/2} \leq \ell^{-1-2\alpha}$, and then the sum is done straightforwardly as

$$\sum_{\mathcal{C}^c} F(x,y) \leq \ell^{-1-2\alpha} \sum_{x,y=0}^{\ell^2-1} h(x, 2\ell^2 - y) (2\ell^2 - x - y - 1)^{-\alpha} \quad (8.66)$$

$$\leq \ell^{-1-2\alpha} \sum_{x=0}^{\ell^2-1} (\ell^2 - x)^{-\alpha} \leq \ell^{1-4\alpha} \quad (8.67)$$

The sum over $n = 1, \dots, n^*$ yields

$$\sum_{x,y=0}^{\ell^2-1} F(x,y) \leq \ell^{1-4\alpha} + \ell^{-2\alpha} \sum_{n \geq 1}^{n^*} (2^n)^{1-1/2-\alpha} \quad (8.68)$$

$$\leq C\ell^{-2\alpha} (2^{1/2-\alpha})^{n^*} \leq C\ell^{1-4\alpha} \quad (8.69)$$

Let us now look at the case where one pair of τ'_1, τ_2, τ'_3 , is consecutive. If $1 + \tau'_1 < \tau'_2 = \tau'_3 - 1$, then we get a factor $\ell^{-2\alpha}$ from $(1 + \tau_2 - \tau_1)^\alpha$ by change of variables, we integrate τ_1 by using the normalization of $h(\cdot)$ and we can treat the remaining times τ'_2, τ'_3 as outlined in the case $|A'| = 2$, gaining an extra factor $\ell^{1-2\alpha}$. The other possible case with one consecutive pair, i.e. $1 + \tau'_1 = \tau'_2 < \tau'_3 - 1$, can be related to the previous case by symmetry considerations, using the translation invariance of $h(\cdot)$. Finally, the case where no pair is consecutive is of course analogous to the corresponding case with $|A'| = 2$: One gets $\ell^{-4\alpha}$ from change of variables, and the remaining sum over $\tau_{1,2,3}$ yields ℓ by using the normalization of $h(\cdot)$ and the factor $(1 + \max I_{\tau'_3} - \tau_3)^{-1/2}$.

Hence in all subcases with $|A'| = 3$, we obtain a factor $\ell^{1-4\alpha}$.

8.3.3 The case $|A'| = k > 3$

We perform the sum over $\tau_1, \tau_2, \dots, \tau_{k-3}$. As argued in (8.49), we get at least one small factor $\ell^{-2\alpha}$ from every second sum. This yields the bound $\ell^{-2\alpha \lfloor \frac{|A'|-3}{2} \rfloor}$. The last three sums are then bounded by $C\ell^{1-4\alpha}$ by repeating the analysis of the case $|A'| = 3$.

Combining all cases, we get the claim of Proposition 8.1

8.4 Initial time linear RG: Proof of Proposition 8.2

We start as in Section 8.3 from the bound (8.50). Now we perform the sum over times τ_1, \dots, τ_k starting from τ_k , keeping in mind that $\tau_1 = 0$.

In the case $k = |A'| > 2$, we get a factor $\ell^{-2\alpha \lfloor \frac{k-1}{2} \rfloor}$ by the change of variables, and the sum is performed by using the summability of the function $h(\tau_1, \dots, \tau_k)$ with $\tau_1 = 0$ fixed. In contrast to the proofs above, this suffices since τ_1 is indeed fixed here: $\tau_1 = 0$. For the case $k = |A'| = 2$, we perform the sum over τ_2 by splitting it in the regions $\tau_2 \geq \ell^2/2$ and $\tau_2 < \ell^2/2$. For $\tau_2 \geq \ell^2/2$, we can use the change of variables, and hence we get

$$\begin{aligned} (8.50) \quad &\leq C \sum_{\tau_2 \geq \ell^2/2} \ell^{-2\alpha} h(0, \tau_2) + C \sum_{1 \leq \tau_2 < \ell^2/2} (\ell^2 - \tau_2)^{-1/2} (1 + \tau_2)^{-\alpha} h(0, \tau_2) \\ &\leq C \ell^{-2\alpha} + C \ell^{-1} \end{aligned} \quad (8.70)$$

This finishes the proof.

8.5 Equilibrium linear RG: Proof of Proposition 8.3

We have to perform the sum

$$\sum_{\tau'_1: \tau'_1 < \tau'_2} \text{dist}(\tau'_1, \tau'_2)^\alpha \|G_{\{\tau'_1, \tau'_2\}, \text{lin}}^{c'}\|_{10\gamma_0} \quad (8.71)$$

Let us abbreviate

$$m_+ = \max I_{\tau'_2} - \tau_2, \quad m_- = \tau_1 - \min I_{\tau'_1} \quad (8.72)$$

Then we obtain

$$\begin{aligned} \|G_{\{\tau'_2, \tau'_2\}, \text{lin}}^{c'}\|_{10\gamma_0} &\leq \sum_{\{\tau_1, \tau_2\} \rightarrow \{\tau'_1, \tau'_2\}} C \frac{\log(1 + m_+)}{(1 + m_+)^{1/2}} \left(\frac{\log(1 + m_-)}{(1 + m_-)^{1/2}} \|G_{\{\tau_1, \tau_2\}}^c\|_{\gamma_0} \right. \\ &\quad \left. + \|G_{\{0, 1, \tau_2 - \tau_1 + 1\}}^c\|_{\gamma_0} + \|G_{\{0, \tau_2 - \tau_1 + 1\}}^c\|_{\gamma_0} + \|G_{\{0, \tau_2 - \tau_1\}}^c\|_{\gamma_0} \right) \end{aligned} \quad (8.73)$$

where we used the time-translation invariance property $G_{\{\tau_1, \tau_2\}}^c = G_{\{1, \tau_2 - \tau_1 + 1\}}^c$, the bounds (8.29) and (8.30), and the bound (6.12) for $\|P^\beta - R(0)\|_{\mathcal{G}}$.

Next, we perform the sum over τ'_1 of the RHS in (8.73). This RHS is split in four terms. The τ'_1 -sum of the last three terms is bounded brutally by $C \ell^2 \epsilon_n^2 \epsilon_{1,n}$ by using the integrability of the function $h(\cdot)$ to perform the sum over $\tau_2 - \tau_1$ and estimating the sum over τ_1 by ℓ^2 . By the smallness of $\epsilon_{1,n}$, this bound is sufficient for Proposition 8.3. Next, we focus on the (τ'_1 -sum of the) first term in (8.73). It is of the form

$$\sum_{\tau'_1: \tau'_1 < \tau'_2} \text{dist}(\tau'_1, \tau'_2)^\alpha \sum_{\{\tau_1, \tau_2\} \rightarrow \{\tau'_1, \tau'_2\}} C (1 + m_+)^{-1/2} (1 + m_-)^{-1/2} \text{dist}(\tau_1, \tau_2)^{-\alpha} h(\tau_1, \tau_2) \quad (8.74)$$

To evaluate this sum, let us first consider the case $\tau'_1 < \tau'_2 - 1$. We set

$$z := \tau_2 - \tau_1, \quad x = \max I_{\tau'_2} - \tau_2 \quad (8.75)$$

and we estimate (8.74) restricted to $\tau'_1 < \tau'_2 - 1$ by

$$C \ell^{-2\alpha} \sum_{z=\ell^2}^{\infty} h(\tau_1, \tau_1 + z) \sum_{x=0}^{\ell^2-1} (1+x)^{-1/2} (1+(x+z) \bmod \ell^2)^{-1/2} \leq C \ell^{-2\alpha} \log \ell \quad (8.76)$$

We used the change of variables to get the factor $\ell^{-2\alpha}$ and the Cauchy-Schwarz inequality to estimate the x -sum. Restricting (8.74) to $\tau'_1 = \tau'_2 - 1$ and setting $y := \tau_1 - \min I_{\tau'_1}$, we get

$$C \sum_{x, y=0}^{\ell^2-1} h(1, \ell^2 - x - y) (1+x)^{-1/2} (1+y)^{-1/2} (\ell^2 - x - y - 1)^{-2\alpha} \quad (8.77)$$

This sum is analogous to the one treated in Section 8.3.2, the only difference being that one exponent is $1/2$ instead of α . The multiscale treatment can be copied without changes. The result is that the sum is bounded by $C \ell^{-2\alpha}$ and this yields Proposition 8.3.

9 Flow of G_A^c : nonlinear part

In this section, no information on T_n is needed, except for the bound on T_n^m from Lemma 7.1. Starting from the induction hypothesis on the cumulants at scale n , we deduce an estimate on the nonlinear part of the contribution to scale $n+1$. We do not distinguish between bulk and boundary terms, and hence we have $A \subset \mathbb{N}_0$ throughout. We drop the scale subscript n and we mark operators on scale $n+1$ by a prime, as explained at the beginning of Section 8.

As was explained in Section 8 as well, the nonlinear contribution to the RG flow is defined by excluding from the sum (3.34) the contributions of $\mathcal{A} = \{A\}$ (a single set) with $|A| = |A'|$. Hence we still need to study the remaining terms

$$G_{A,\text{nl}}^{c'} := \mathcal{S}_\ell \sum_{\substack{\mathcal{A} \in \mathfrak{B}(I_{A'}), \mathcal{G}_{A'}(\mathcal{A}) \text{ connected} \\ \sum_{A \in \mathcal{A}} |A| > |A'|}} \mathcal{T}_{A'} \left[\bigotimes_{A \in \mathcal{A}} G_A^c \otimes_{\tau \notin \text{Supp} \mathcal{A}} T(\tau) \right] \quad (9.1)$$

that is, either \mathcal{A} has more than one element, or if it consists of one element A , then $|A| > |A'|$, and this is combined in the condition $\sum_{A \in \mathcal{A}} |A| > |A'|$. Our result is

Proposition 9.1. *Fix an exponent $\hat{\alpha}$ with $\hat{\alpha} < \alpha$ and a constant $c_v > 0$. If c_v is chosen small enough, then*

$$\sup_{\tau'} \sum_{A' \subset \mathbb{N}_0: A' \ni \tau'} (\hat{\epsilon}')^{-|A'|} \text{dist}(A')^\alpha \|G_{A',\text{nl}}^{c'}\|_{10\gamma_0} \leq 1, \quad \text{with } \hat{\epsilon}' = c_v^{-1} \ell^{-\hat{\alpha}} \epsilon, \quad (9.2)$$

We remind the conventions introduced at the beginning of Section 6.1; ℓ^{-1} and ϵ_0 should be chosen sufficiently small compared to constants like c_v . Proposition 9.1 establishes (the nonlinear parts of) Proposition 6.2. Indeed, for $\alpha > 1/2$, it suffices to fix $\tilde{\alpha} < \hat{\alpha} < \alpha$ whereas for the case $\alpha \leq 1/2$, we observe that we can choose $\hat{\alpha}$ such that additionally $k\tilde{\alpha} + \tilde{\alpha}_1 < \tilde{k}\hat{\alpha}$ holds for $k \geq 2$.

To derive Proposition 9.1, one can ignore the Ward Identities completely since they can at best give a factor ℓ^2 in each term of the sum, and such factors are irrelevant in view of the fact that we have extra factors of ϵ due to the condition $\sum_{A \in \mathcal{A}} |A| > |A'|$ (cfr. the discussion at the beginning of Section 8).

Consequently, we will use the following basic bound for the cumulants. In fact, a special case of this bound appeared already in the proof of Lemma 7.2.

Lemma 9.1 (A priori recursion relation).

$$\|G_{A'}^{c'}\|_{10\gamma_0} \leq \sum_{\mathcal{A} \in \mathfrak{B}(I_{A'}), \mathcal{G}_{A'}(\mathcal{A}) \text{ connected}} e^{C|\text{Supp} \mathcal{A}|} \prod_{A \in \mathcal{A}} \|G_A^c\|_{\gamma_0} \quad (9.3)$$

Proof. We take $\ell > 10$ and apply Lemma 5.3 to eq. (3.34):

$$\left\| \mathcal{T}_{\tau'} \left[\bigotimes_{A \in \mathcal{A}} G_A^c \otimes_{\tau \notin \text{Supp} \mathcal{A}} T(\tau) \right] \right\|_{10\gamma_0} \leq \prod_{A \in \mathcal{A}} \|G_A^c\|_{\gamma_0} \prod_J \|T^{|J|}\|_{\gamma_0} \quad (9.4)$$

where the product \prod_J runs over all discrete intervals J in the sets $I_{\tau'} \setminus \text{Supp} \mathcal{A}$, cfr. the proof of Lemma 7.2. By invoking Lemma 7.1, we bound

$$\|T^{|J|}\|_{\gamma_0} \leq C, \quad \text{since } |J| \leq \ell^2 \quad (9.5)$$

The number of discrete intervals J is at most $1 + |\text{Supp} \mathcal{A}| < 2|\text{Supp} \mathcal{A}|$ and this yields the claim. \square

9.1 Sum over connected coverings

Our strategy for bounding the sum over polymers \mathcal{A} in eq. (9.1) consists of the following splitting. For each $A \in \mathcal{A}$, we let $S(A) \subset A'$ be the macroscopic domain of A , that is

$$S(A) = \{\tau' \in A', A \cap I_{\tau'} \neq \emptyset\} \quad (9.6)$$

Note that any collection \mathcal{A} of sets A induces a collection $\mathcal{S} = \mathcal{S}(\mathcal{A})$ with elements $S(A)$. We call a collection of sets *connected* whenever it can not be split into two collections whose members are mutually disjoint. For any $\mathcal{A} \in \mathfrak{B}(I_{A'})$, the connectedness of the graph $\mathcal{G}_{A'}(\mathcal{A})$ implies that $\mathcal{S} = \mathcal{S}(\mathcal{A})$ is connected and that $\text{Supp}\mathcal{S} = A'$. We call the set of connected collections \mathfrak{C} .

With this terminology, (9.3) can be written as

$$\|G_{A'}^{c'}\|_{10\gamma_0} \leq \sum_{\mathcal{S} \in \mathfrak{C}, \text{Supp}\mathcal{S}=A'} \sum_{\mathcal{A} \in \mathfrak{B}(I_{A'}): \mathcal{S}(A)=\mathcal{S}} \prod_{A \in \mathcal{A}} \|G_A^c\|_{\gamma_0} e^{C|A|} \quad (9.7)$$

$$\leq \sum_{\mathcal{S} \in \mathfrak{C}, \text{Supp}\mathcal{S}=A'} \prod_{S \in \mathcal{S}} F_1 \left(\sum_{A \rightarrow S} \|G_A^c\|_{\gamma_0} e^{C|A|} \right) \quad (9.8)$$

where the function $F_1(x) := \sum_{p=1}^{\infty} x^p = x/(1-x)$ appears because there can be more than one set $A \in \mathcal{A}$ such that $A \rightarrow S$.

Let us now determine how this bound can be modified if we restrict the sum in (9.3) to those entering in the nonlinear RG, i.e. in (9.1). If $|\mathcal{S}| > 2$, then the condition $\sum_{A \in \mathcal{A}} |A| > |A'|$ is always verified. If $|\mathcal{S}| = 1$, i.e. there is one $S = A'$, then there are either at least two $A \in \mathcal{A}$ such that $A \rightarrow A'$, or there is only one but it satisfies $|A| > |A'|$. Hence we get

$$\|G_{A'}^{c'}\|_{10\gamma_0} \leq \sum_{\mathcal{S} \in \mathfrak{C}, \text{Supp}\mathcal{S}=A', |\mathcal{S}| > 1} \prod_{S \in \mathcal{S}} F_1 \left(\sum_{A \rightarrow S} \|G_A^c\|_{\gamma_0} e^{C|A|} \right) \quad (9.9)$$

$$+ \sum_{A \rightarrow A', |A| > |A'|} \|G_A^c\|_{\gamma_0} e^{C|A|} + F_2 \left(\sum_{A \rightarrow A'} \|G_A^c\|_{\gamma_0} e^{C|A|} \right) \quad (9.10)$$

where $F_2(x) = \sum_{p=2}^{\infty} x^p$.

The following lemma shows how to control sums over $A \rightarrow S$ that will appear in the evaluation of the above expression

Lemma 9.2. 1)

$$\sum_{A \rightarrow S} e^{C|A|} \text{dist}(A)^\alpha \|G_A^c\|_{\gamma_0} \leq (C\epsilon)^{|\mathcal{S}|} \quad (9.11)$$

2) *Abbreviate*

$$v_n(S) := (\ell^{-\hat{\alpha}}\epsilon)^{-|\mathcal{S}|} \text{dist}(S)^\alpha \sum_{A \rightarrow S} \|G_A^c\|_{\gamma_0} e^{C|A|}. \quad (9.12)$$

Then

$$\sup_{\tau'} \sum_{S \ni \tau'} v_n(S) \leq C\ell^{2+2\alpha}. \quad (9.13)$$

3)

$$\sup_{\tau'} \sum_{S \ni \tau'} (\ell^{-\hat{\alpha}}\epsilon)^{-|\mathcal{S}|} \text{dist}(S)^\alpha \sum_{A \rightarrow S, |A| > |\mathcal{S}|} \|G_A^c\|_{\gamma_0} e^{C|A|} \leq C\ell^{2+3\alpha}\epsilon. \quad (9.14)$$

Proof. To get (9.11), we recall the function $h(\cdot)$ introduced in (8.47), and we bound the LHS by

$$\sum_{k \geq |\mathcal{S}|} (C\epsilon)^k \sum_{\tau_1 \in I_{\tau'_1}} \sum_{A: |A|=k, \min A=\tau_1} h(A) \leq \ell^2 (C\epsilon)^{|\mathcal{S}|}, \quad \tau'_1 = \min S \quad (9.15)$$

using the summability of h , i.e. (8.46) and bounding the sum over $\tau_1 = \min A$ by ℓ^2 .

Now to (9.13); we rewrite

$$\sum_{S \ni \tau'} v_n(S) \leq \sum_{A: A \cap I_{\tau'} \neq \emptyset} (\ell^{-\hat{\alpha}} \epsilon)^{-|S(A)|} \epsilon^{|A|} \text{dist}(S(A))^\alpha (\text{dist}(A))^{-\alpha} h(A) \quad (9.16)$$

Assume there is a $\tau' \in S(A)$, $\tau' \neq \min S(A)$, $\max S(A)$, such that the set $A \cap I_{\tau'}$ contains only one element. Then we get a factor $\ell^{-2\alpha}$ from change of variables, and if there are p such τ' , then we get at least $\lfloor (p+1)/2 \rfloor$ of these factors. From elementary considerations,

$$2(\lfloor (p+1)/2 \rfloor) \geq p \geq |S(A)| - (|A| - |S(A)|) - 2 \quad (9.17)$$

Hence, we bound

$$\begin{aligned} \sum_{S \ni \tau'} v_n(S) &\leq \sum_{A: A \cap I_{\tau'} \neq \emptyset} (\ell^{-\hat{\alpha}} \epsilon)^{-|S(A)|} \epsilon^{|A|} \ell^{-\alpha(2|S(A)| - |A| - 2)} h(A) \\ &\leq \ell^{2\alpha} \sum_{k_A \geq k_S \geq 2} (\epsilon \ell^\alpha)^{k_A - k_S} \ell^{(\hat{\alpha} - \alpha)k_S} \sum_{A: A \cap I_{\tau'} \neq \emptyset, |A| = k_A} h(A) \\ &\leq \ell^{2\alpha+2} \sum_{k_A \geq k_S \geq 2} k_A (\epsilon \ell^\alpha)^{k_A - k_S} \ell^{(\hat{\alpha} - \alpha)k_S} \leq C \ell^{2+2\alpha} \end{aligned} \quad (9.18)$$

The second inequality follows by setting $k_S = |S(A)|$, $k_A = |A|$. To get the third inequality, we bound the sum over $h(A)$ with $A \ni \tau$ by $|A|$ times the sum over $h(A)$ with $\min A$ fixed, and then we proceed as in the proof of, (9.11). This yields (9.13). To obtain (9.14), one repeats the calculation with the only difference that the sum over k_A, k_S in (9.18) is restricted to $k_A - 1 \geq k_S \geq 2$. \square

9.2 Proof of Proposition 9.1

We perform the sum

$$\sum_{A' \ni \tau'} (\epsilon')^{-|A'|} \text{dist}(A')^\alpha \|G_{A', \text{nlm}}^{c'}\|_{10\gamma_0} \leq \mathbf{I} + \mathbf{II} + \mathbf{III} \quad (9.19)$$

where the three terms on the RHS refer to the three terms in (9.10). The second term can be bounded immediately with the help of (9.14) (with $S = A'$), yielding

$$\mathbf{II} \leq C \ell^{2+3\alpha} \epsilon \quad (9.20)$$

We estimate the term \mathbf{III} , with $x := \sum_{A \rightarrow A'} \|G_A^c\|_{\gamma_0} e^{C|A|} \leq (C\epsilon)^{|A'|}$ by (9.11) in Lemma 9.2. In particular, we have $x < 1 - c$, and hence $F_2(x) \leq \epsilon^2 C x$ with F_2 as in (9.10), since $|A'| \geq 2$. Therefore,

$$\mathbf{III} \leq C \epsilon^2 \sum_{A' \ni \tau'} (\epsilon')^{-|A'|} \text{dist}(A')^\alpha \sum_{A \rightarrow A'} \|G_A^c\|_{\gamma_0} e^{C|A|} \leq \epsilon^2 \ell^{2+2\alpha} \quad (9.21)$$

where we used (9.13) in Lemma 9.2 (recognizing the definition of $v_n(S)$ with $S = A'$). Now we turn to

$$\mathbf{I} \leq \sum_{A' \ni \tau'} (\epsilon')^{-|A'|} \text{dist}(A')^\alpha \sum_{S \in \mathfrak{C}, \text{Supp } S = A', |S| > 1} \prod_{S \in \mathcal{S}} \sum_{A \rightarrow S} \|G_A^c\|_{\gamma_0} e^{C|A|} \quad (9.22)$$

where we dropped the function $F_1(\cdot)$ at the cost of increasing the constant C , using, as in the treatment of term \mathbf{III} above, that its argument is smaller than $1 - c$. First, for any $S \in \mathfrak{C}$ with $\text{Supp } S = A'$, Lemma B.1 implies

$$\text{dist}(A')^\alpha \leq \prod_{S \in \mathcal{S}} \text{dist}(S)^\alpha \quad (9.23)$$

such that (9.22) is dominated by

$$\sum_{S \in \mathfrak{C}, |\mathcal{S}| > 1, \text{Supp } S \ni \tau'} (\epsilon')^{-|\text{Supp } S|} \prod_{S \in \mathcal{S}} \text{dist}(S)^\alpha \left(\sum_{A \rightarrow S} \|G_A^c\|_{\gamma_0} e^{C|A|} \right) \quad (9.24)$$

By substituting (9.12) for each $S \in \mathcal{S}$ in (9.24), and setting $N(\mathcal{S}) = \sum_{S \in \mathcal{S}} |S|$,

$$(9.22) \leq \sum_{S \in \mathfrak{C}, |\mathcal{S}| > 1, \text{Supp } S \ni \tau'} \underbrace{(\epsilon')^{N(\mathcal{S}) - |\text{Supp } \mathcal{S}|} (\ell)^{(2+2\alpha)|\mathcal{S}|}}_{r(\mathcal{S})} \prod_{S \in \mathcal{S}} c_v^{|S|} \ell^{-(2+2\alpha)|S|} v_n(S) \quad (9.25)$$

By the connectedness of \mathcal{S} , we have $N(\mathcal{S}) - |\text{Supp } \mathcal{S}| \geq |\mathcal{S}| - 1$, and combined with $|\mathcal{S}| \geq 2$, this yields $r(\mathcal{S}) \leq 1$. Hence we are left with the task of estimating (the constraint $|\mathcal{S}| > 1$ can now be dropped)

$$\sum_{S \in \mathfrak{C}, \text{Supp } S \ni \tau'} \prod_{S \in \mathcal{S}} c_v^{|S|} \ell^{-(2+2\alpha)|S|} v_n(S). \quad (9.26)$$

The tool to perform this sum is the bound $\sum_{S \ni \tau'} \ell^{-(2+2\alpha)|S|} v_n(S) \leq C$ from Lemma 9.2. Indeed, sums over collections of connected sets, as in (9.26), can be handled conveniently with cluster expansions, with the bound from Lemma 9.2 playing the role of the Kotecky-Preiss criterion. We state the relevant cluster expansion result in Proposition B.1 in Appendix B. A basic corollary, eq. (B5), implies that (9.26) is bounded by a constant that can be made smaller than 1 by choosing c_v small enough.

Collecting the bounds on the three terms **I, II, III**, Proposition 9.1 follows

10 Estimates on the first scale: Excitations

In this section, we prove bounds on the correlation functions G_A^c , i.e. the claims of Induction hypothesis 6.2 for $n = 0$. We treat the bulk correlation functions ($0 \ni A$) in Section 10.2 and the boundary correlation functions ($0 \in A$) in Section 10.3. We derive an explicit representation for the operators G_A^c that is based on the Dyson expansion. This representation will also be useful to analyze T for $n = 0$ in Section 11. Since we start so to say from scratch, the first part of our discussion is in finite volume Λ and we perform the thermodynamic limit in Section 10.2.1.

10.1 Dyson Expansion

We recall the reduced dynamics of the system at microscopic time t

$$Z_t \rho_S = \text{Tr}_E [e^{-itH} (\rho_S \otimes \rho_E^{\text{ref}}) e^{itH}] \quad (10.1)$$

and the dynamics on the $n = 0$ scale, $T_{n=0}$ (introduced in Section 3), it is related to Z_t through $T_0 = Z_{\lambda^{-2}t_0}$. In the next sections, we will develop an expansion for Z_t and we will relate this expansion to the correlation functions G_A^c .

10.1.1 Derivation of the expansion

We introduce the time-evolved interaction Hamiltonian

$$H_I(s) := e^{isH_E} \sum_{q \in \Lambda^*, i=1,2} (\phi(q)(W \otimes e^{iqX} \otimes a_{i,q}^*) + h.c.) e^{-isH_E} \quad (10.2)$$

and define the Liouvillians $L_I(s) = \text{ad}(H_I(s))$, $L_S = \text{ad}(H_S)$ and $L_E = \text{ad}(H_E)$. Let us also use the shorthand $U_s = e^{-isL_S}$. Then the Duhamel formula

$$e^{itL_E} e^{-itL} = U_t + \int_0^t ds U_{t-s} (-i\lambda L_I(s)) e^{isL_E} e^{-isL}$$

yields upon iteration the Dyson series for the reduced dynamics Z_t :

$$Z_t \rho_S = \sum_{m \geq 0} (-\lambda^2)^m \int_{0 < t_1 < \dots < t_{2m} < t} dt_1 \dots dt_{2m} \operatorname{Tr}_E \left[U_{t-t_{2m}} L_I(t_{2m}) \dots U_{t_2-t_1} L_I(t_1) U_{t_1} (\rho_S \otimes \rho_E^{\text{ref}}) \right] \quad (10.3)$$

where the invariance $\operatorname{Tr}_E(e^{-itL_E} A) = \operatorname{Tr}_E A$ was used. For $m = 0$, the RHS is understood as $U_t \rho_S$.

Next, we write the integrand using the formalism developed in Section 3.1:

$$\operatorname{Tr}_E \left[U_{t-t_{2m}} L_I(t_{2m}) \dots U_{t_2-t_1} L_I(t_1) U_{t_1} (\rho_S \otimes \rho_E^{\text{ref}}) \right] \quad (10.4)$$

$$= \mathcal{T} \mathbb{E} [U_{t-t_{2m}} \otimes_S L_I(t_{2m}) \otimes_S \dots \otimes_S U_{t_2-t_1} \otimes_S L_I(t_1) \otimes_S U_{t_1}] \rho_S. \quad (10.5)$$

We recall that the expectation \mathbb{E} acts on an element of $\mathcal{R}^{\otimes n} \otimes \mathcal{R}_E$ and the contraction $\mathcal{T} : \mathcal{R}^{\otimes n} \rightarrow \mathcal{R}$. As in Section 3.1, it will be convenient to label the spaces \mathcal{R} in $\mathcal{R}^{\otimes n}$ by the times that occur in the corresponding operators. Given a $2m$ -tuple of times $\{t_1, \dots, t_{2m}\} \equiv \underline{t} \subset [0, t]$ let $t_0 = 0, t_{2m+1} = t$ and denote the family of intervals $J_i \equiv [t_i, t_{i+1}]$, $i = 0, 2m$ by $\mathcal{J}(\underline{t})$. We will then index the space \mathcal{R} where $L_I(t_i)$ lies by \mathcal{R}_{t_i} and the space \mathcal{R} where $U_{t_{i+1}-t_i}$ lies by \mathcal{R}_{J_i} . Thus

$$\mathbb{E} [U_{t-t_{2m}} \otimes_S L_I(t_{2m}) \otimes_S \dots \otimes_S U_{t_2-t_1} \otimes_S L_I(t_1) \otimes_S U_{t_1}] \in \bigotimes_{i=1}^{2m} \mathcal{R}_{t_i} \otimes_{J \in \mathcal{J}(\underline{t})} \mathcal{R}_J \quad (10.6)$$

and as before the operator \mathcal{T} contracts the operators in the obvious chronological order, i.e. such that those in $\mathcal{R}_{[0, t_1]}$ are on the right, then those in \mathcal{R}_{t_1} , then those in $\mathcal{R}_{[t_1, t_2]}$, etc.

Let $\{u, v\}$ be a pair of (distinct) times with the convention that $u < v$. Then we define

$$K_{u,v} = -\lambda^2 \mathbb{E} [L_I(v) \otimes_S L_I(u)], \quad K_{u,v} \in \mathcal{R} \otimes \mathcal{R} \quad (10.7)$$

and we view this operator as an element in $\mathcal{R}_v \otimes \mathcal{R}_u$. We also abbreviate $U_{s'-s}$ by U_J with $J = [s, s']$ and we view U_J as an element in \mathcal{R}_J . Since L_I is linear in the creation and annihilation operators, the Wick theorem yields

$$\mathbb{E} [U_{t-t_{2m}} \otimes_S L_I(t_{2m}) \otimes_S \dots \otimes_S U_{t_2-t_1} \otimes_S L_I(t_1) \otimes_S U_{t_1}] = \sum_{\pi \in \text{Pairings}(\underline{t})} \bigotimes_{\{u,v\} \in \pi} K_{u,v} \otimes_{J \in \mathcal{J}(\underline{t})} U_J \quad (10.8)$$

where the sum on the RHS runs over pairings π , i.e. partitions of the times t_1, \dots, t_{2m} in m pairs (u, v) with the convention that $u < v$.

By plugging (10.8) into (10.3), we obtain

$$Z_t = \sum_{m \geq 0} \int_{0 < t_1 < \dots < t_{2m} < t} dt_1 \dots dt_{2m} \sum_{\pi \in \text{Pairings}(\underline{t})} \mathcal{T} [\bigotimes_{\{u,v\} \in \pi} K_{u,v} \otimes_{J \in \mathcal{J}(\underline{t})} U_J] \quad (10.9)$$

As before, we equip products of \mathcal{R} , e.g. as in (10.6), with the norm $\|\cdot\|_\gamma$.

10.1.2 A formalism for the combinatorics

The integral over ordered \underline{t} , together with the sum over pairings, π , on the set of times, is represented as a combined integral and sum over ordered pairs (u_i, v_i) with $u_i, v_i \in \mathbb{R}^+$ and $i = 1, \dots, m$, such that

$$u_i < v_i, \quad u_1 < \dots < u_m \quad (10.10)$$

This is done as follows. For any pair $(r, s) \in \pi$, we let $u_i = t_r, v_i = t_s$ where the index $i = 1, \dots, m$ is chosen such that the u_i are ordered $u_1 < u_2 < \dots < u_m$. We represent one pair (u_i, v_i) by the symbol w_i and the m -tuple of them by \underline{w} . We call Ω_J the set of \underline{w} such that $u_i, v_i \in J$ (for arbitrary m), and we use the shorthand

$$\int_{\Omega_J} d\underline{w} := \sum_{m \geq 0} \int_{J^m} d\underline{u} \int_{J^m} d\underline{v} 1_{[u_i < v_i]} 1_{[u_1 < \dots < u_m]} \quad (10.11)$$

where the RHS is set to 1 for $m = 0$, corresponding to $\underline{w} = \emptyset$ in the LHS. In what follows, we will often consider the ordered times $\underline{t}, \underline{u}, \underline{v}$ to be implicitly defined by \underline{w} . For example, we will write $\mathcal{J}(\underline{w})$ instead of $\mathcal{J}(\underline{t})$. The Dyson expansion in terms of the sets of pairs \underline{w} reads

$$Z_t = \int_{\Omega_{[0,t]}} d\underline{w} \mathcal{T} \left[\bigotimes_{w \in \underline{w}} K_w \bigotimes_{J \in \mathcal{J}(\underline{w})} U_J \right] \quad (10.12)$$

10.1.3 Connected correlations and the Dyson series

To relate the previous sections to the setup in Section 3, we need to discretize time and express the operators G_A, G_A^c in terms of the Dyson series.

Recall that \mathbb{N} is the set of macroscopic times. To each macroscopic time, we now associate a *domain* of microscopic times,

$$\text{Dom}(\tau) = [\lambda^{-2}t_0(\tau - 1), \lambda^{-2}t_0\tau] \quad (10.13)$$

To a set $A \subset \mathbb{N}$ of macroscopic times, we then associate the domain

$$\text{Dom}(A) = \bigcup_{\tau \in A} \text{Dom}(\tau) \quad (10.14)$$

We take $t = \lambda^{-2}t_0N$. Then, a set of pairs $\underline{w} \in \Omega_{[0,t]}$ determines a graph $\mathcal{G}(\underline{w})$ on $\{1, \dots, N\}$ by the following prescription: the vertices $\tau < \tau'$ are connected by an edge if and only if there is a pair $w = (u, v)$ in \underline{w} such that

$$u \in \text{Dom}(\tau) \quad \text{and} \quad v \in \text{Dom}(\tau') \quad (10.15)$$

(Note that there may be several such pairs). We write $\text{Supp}(\mathcal{G}(\underline{w}))$ for the set of non-isolated vertices of $\mathcal{G}(\underline{w})$, i.e. the vertices that have at least one connection to another vertex. If $\underline{w} \in \Omega_{\text{Dom}A}$, then $\text{Supp}(\mathcal{G}(\underline{w}))$ is obviously a subset of A . In that case we write $\mathcal{G}_A(\underline{w})$ for the induced graph with vertex set A . The graphs $\mathcal{G}(\underline{w})$ in the Dyson expansion with support A give rise to the correlation function G_A of Section 3, and connected graphs $\mathcal{G}_A(\underline{w})$ give rise to the connected correlation functions. This goes as follows. Recall the collection $\mathcal{J}(\underline{w})$ of intervals determined by the times in \underline{w} . Given a macroscopic time $\tau \in \mathbb{N}$ we define $\mathcal{J}_\tau(\underline{w})$ as the family of intervals $\{\text{Dom}(\tau) \cap J \mid J \in \mathcal{J}(\underline{w})\}$ and set

$$\mathcal{J}_A(\underline{w}) := \bigcup_{\tau \in A} \mathcal{J}_\tau(\underline{w}) \quad (10.16)$$

Using the group property of U_s and the definition of contraction we have

$$U_J = \iota(U_{J_2} \otimes U_{J_1})$$

for $J = J_1 \cup J_2$ with J_1, J_2 consecutive intervals. Applying this to the intervals J in (10.12) intersecting more than one $\text{Dom}(\tau)$ we get

$$\mathcal{T} \left[\bigotimes_{w \in \underline{w}} K_w \bigotimes_{J \in \mathcal{J}(\underline{w})} U_J \right] = \mathcal{T} \left[\bigotimes_{w \in \underline{w}} K_w \bigotimes_{J \in \mathcal{J}_{[1,N]}(\underline{w})} U_J \right] =: \mathcal{TV}_{[0,N]}(\underline{w}). \quad (10.17)$$

where we abbreviated $[1, N] = \{1, 2, \dots, N\}$. The tensor product defining $\mathcal{V}_{[1,N]}(\underline{w})$ factors across the macroscopic times, i.e:

$$\mathcal{V}_{[1,N]}(\underline{w}) = \bigotimes_{\tau \notin \text{Supp}(\mathcal{G}(\underline{w}))} \mathcal{V}_{\text{Dom}(\tau)}(\underline{w}) \otimes \mathcal{V}_{\text{Supp}(\mathcal{G}(\underline{w}))}(\underline{w}) \quad (10.18)$$

which can also be written as

$$\mathcal{V}_{[1,N]}(\underline{w}) = \bigotimes_{\tau \notin \text{Supp}(\mathcal{G}(\underline{w}))} \mathcal{V}_{\text{Dom}(\tau)}(\underline{w}) \otimes_i \mathcal{V}_{\text{Supp}(\mathcal{G}_i(\underline{w}))}(\underline{w}) \quad (10.19)$$

where \mathcal{G}_i are the connected components of \mathcal{G} .

As in Section (3.5), we may perform the time ordered contraction \mathcal{T} in two steps, first within the time intervals $\text{Dom}(\tau)$ and then contracting the rest:

$$\mathcal{T} = \mathcal{T} \bigotimes_{\tau=1}^N \mathcal{T}_\tau \quad (10.20)$$

where

$$\mathcal{T}_\tau : \bigotimes_{i:t_i \in \text{Dom}(\tau)} \mathcal{R}_{t_i} \bigotimes_{J \in \mathcal{J}_\tau(\underline{w})} \mathcal{R}_J \rightarrow \mathcal{R}_\tau.$$

The beautiful formula (10.20) is not a typo but a consequence of the fact that we defined \mathcal{T} both as contracting microscopic times and intervals, and macroscopic intervals. Writing as before $\mathcal{T}_A = \bigotimes_{\tau \in A} \mathcal{T}_\tau$, eq. (10.18) leads then to the expansion (3.6) with

$$G_A = \int_{\Omega_{\text{Dom}(A)}} d\underline{w} \, 1_{[\text{Supp}(\mathcal{G}(\underline{w}))=A]} \mathcal{T}_A \left[\bigotimes_{w \in \underline{w}} K_w \bigotimes_{J \in \mathcal{J}_A(\underline{w})} U_J \right] \quad (10.21)$$

and

$$T = \int_{\Omega_{\text{Dom}(\tau)}} d\underline{w} \, \mathcal{T} \left[\bigotimes_{w \in \underline{w}} K_w \bigotimes_{J \in \mathcal{J}_\tau(\underline{w})} U_J \right] \quad (10.22)$$

(note that τ on the RHS is arbitrary) whereas (10.19) gives

$$G_A^c = \int_{\Omega_{\text{Dom}(A)}} d\underline{w} \, 1_{[\mathcal{G}_A(\underline{w}) \text{ connected}]} \mathcal{T}_A \left[\bigotimes_{w \in \underline{w}} K_w \bigotimes_{J \in \mathcal{J}_A(\underline{w})} U_J \right] \quad (10.23)$$

10.2 Bounds on bulk correlation functions

In this Section, we state and prove Lemma 10.1, which is actually the induction hypothesis 6.2 for bulk sets A (not containing 0) and on scale $n = 0$. Boundary correlation functions ($A \ni 0$) will be treated in Section 10.3.

10.2.1 Bounds on the Dyson expansion

As a first step, let us derive a term by term bound on the Dyson expansion and perform the thermodynamic limit. Let us write the operator $K_{u,v}$ in (10.7) explicitly. Using the field operators (2.15) we can rewrite (10.2) as

$$H_I(s) = \sum_x W \otimes \mathbb{1}_x \otimes \Phi(x, s). \quad (10.24)$$

Let us use the notation $\text{ad}(A) = A^0 - A^1$ where A^0 is left multiplication and A^1 is right multiplication (by A). Then, (10.7) becomes

$$K_{u,v} = -\lambda^2 \sum_{x,y} \sum_{a,b \in \{0,1\}} (-1)^{a+b} \zeta^{ab}(x-y, v-u) (W \otimes \mathbb{1}_x)^a \otimes (W \otimes \mathbb{1}_y)^b \quad (10.25)$$

where

$$\zeta^{ab}(x-y, v-u) = \text{Tr}_E \left(\Phi(x, v)^a \Phi(y, u)^b \rho_E^{\text{ref}} \right) \quad (10.26)$$

By (2.16)

$$\zeta^{00}(x, t) = \zeta^{10}(x, t) = \zeta^{11}(x, -t) = \zeta^{01}(x, -t) = \zeta(x, t) \quad (10.27)$$

(we used translation invariance in time and $O(d)$ invariance in x).

Eq. (2.17) (thermodynamic limit for ζ) implies the kernel of $K_{u,v}$ has a pointwise limit as $\Lambda \nearrow \mathbb{Z}^d$. For the remainder of the present section, we use the notation $K_{u,v}$ and $h(s)$ (introduced below) both for Λ finite and $\Lambda = \mathbb{Z}^d$, indicating differences whenever necessary. Since the kernel of $\mathbb{1}_y^a$ is diagonal in the coordinates x, v (recall

that $z = (x, v, \eta, e_L, e_R) \in \mathbb{A}_0$ and note that this v has nothing to do with the time-coordinates u, v used below), we get

$$\|K_{u,v}\|_\gamma \leq C\lambda^2 \quad (10.28)$$

for all γ . In fact, using the time decay of ζ in assumption A and denoting

$$\lambda^2 h(v - u) := \|K_{u,v}\|_{20\gamma_0}, \quad (10.29)$$

we get $h(s) \leq C$ for Λ finite and

$$\int_0^\infty ds (1 + |s|)^\alpha h(s) \leq C, \quad \text{for } \Lambda = \mathbb{Z}^d \quad (10.30)$$

Lemma 10.1. *The sums and integrals on the RHS of (10.12), (10.21) (10.22) and (10.23) converge absolutely. For example, the series defining Z_t is bounded by*

$$\int_{\Omega_{[0,t]}} d\underline{w} \left\| \mathcal{T} \left[\bigotimes_{w \in \underline{w}} K_w \bigotimes_{J \in \mathcal{J}(\underline{w})} U_J \right] \right\|_{20\gamma_0} \leq \begin{cases} e^{C\lambda^2 t^2} & \Lambda \text{ finite} \\ e^{C\lambda^2 t} & \Lambda = \mathbb{Z}^d \end{cases} \quad (10.31)$$

In particular, the limits of (10.12), (10.21) (10.22) and (10.23) as $\Lambda \nearrow \mathbb{Z}^d$ exist. Moreover Z_t is strongly continuous in t .

Proof. The following reasoning applies for finite and infinite Λ alike. Using Lemma 4.3. we get

$$\left\| \mathcal{T} \left[\bigotimes_{w \in \underline{w}} K_w \bigotimes_{J \in \mathcal{J}(\underline{w})} U_J \right] \right\|_\gamma \leq \prod_{w \in \underline{w}} \|K_w\|_\gamma \times \prod_{J \in \mathcal{J}(\underline{w})} \|U_J\|_\gamma \quad (10.32)$$

and using standard propagation bounds for the lattice Laplacian Δ , we have

$$\|U_J\|_{20\gamma_0} \leq C e^{\lambda^2 C |J|} \quad (10.33)$$

(recall that we treat γ_0 as a constant). Hence the LHS of (10.31) can be bounded by

$$\int_{\Omega} d\underline{w} \prod_{J \in \mathcal{J}_{[0,t]}(\underline{w})} C e^{\lambda^2 C |J|} \times \prod_{w \in \underline{w}} \lambda^2 h(v - u) \quad (10.34)$$

$$\leq e^{\lambda^2 C t} \sum_{m \in \mathbb{N}} \int_{0 < u_1 < \dots < u_m < t} d\underline{u} \left(\prod_{i=1}^m C \int_{u_i}^t dv_i h(v_i - u_i) \right) \quad (10.35)$$

$$\leq e^{C\lambda^2 t} \sum_{m \geq 0} \frac{(\lambda^2 C t)^m}{m!} \left(\int_0^t ds h(s) \right)^m \leq \begin{cases} e^{C\lambda^2 t^2} & \Lambda \text{ finite} \\ e^{C\lambda^2 t} & \Lambda = \mathbb{Z}^d \end{cases} \quad (10.36)$$

To get the second last inequality, we first performed the v_i -integrals, and then we estimated the u_i -integrals by the volume of an m -dimensional simplex. In the last one we used eq. (10.30) and $h(s) > C$ for finite Λ . For any finite t , the term-by-term convergence of the series follows by the Vitali convergence theorem, since each term converges pointwise and the series is summable uniformly in Λ . For the series in (10.21, 10.22, 10.23), similar reasoning applies. Strong continuity of $t \mapsto Z_t$ follows from strong continuity of $s \mapsto U_s$. \square

The following estimate is an immediate consequence of (10.23) and bounds as in the proof of Lemma 10.1.

Lemma 10.2.

$$\|G_A^c\|_{10\gamma_0} \leq e^{C|A|} \int_{\substack{\Omega_{\text{Dom}(A)} \\ \underline{w} \text{ spans } A \text{ minimally}}} d\underline{w} \prod_{w \in \underline{w}} \lambda^2 C h(v - u) \quad (10.37)$$

The statement ‘ \underline{w} spans A minimally’ means that $\mathcal{G}_A(\underline{w})$ is connected and that no pair can be dropped from \underline{w} without losing this property. In particular, this implies that $\mathcal{G}_A(\underline{w})$ is a spanning tree on A .

Proof. By (10.23) and Lemma 10.1, we have

$$\|G_A^c\|_{10\gamma_0} \leq \int_{\substack{\Sigma_{\text{Dom}(A)} \\ \mathcal{G}_A(\underline{w}) \text{ connected}}} d\underline{w} \prod_{J \in \mathcal{J}_A(\underline{w})} C e^{\lambda^2 C |J|} \times \prod_{w \in \underline{w}} \lambda^2 h(v-u) \quad (10.38)$$

and, since $|\mathcal{J}_A(\underline{w})| \leq |\underline{w}| + |A|$, we may dominate the integrand by $e^{C|A|} \prod_{w \in \underline{w}} \lambda^2 C h(v-u)$, since $|\text{Dom}(A)| = \lambda^{-2}|A|$. Next, we state an appealing estimate was the main motivation for encoding the pairings π in the pair-sets \underline{w} . For any (integrable) function f on $\Omega_{\text{Dom}(A)}$, we have

$$\int_{\substack{\Sigma_{\text{Dom}(A)} \\ \mathcal{G}_A(\underline{w}) \text{ connected}}} d\underline{w} |f(\underline{w})| \leq \int_{\substack{\Omega_{\text{Dom}(A)} \\ \underline{w}' \text{ spans } A \text{ minimally}}} d\underline{w}' \int_{\Sigma_{\text{Dom}(A)}} d\underline{w}'' |f(\underline{w}' \cup \underline{w}'')| \quad (10.39)$$

To realize why this holds true, choose a spanning tree \mathcal{T} for the connected graph $\mathcal{G}_A(\underline{w})$ and then pick a minimal subset \underline{w}' of the pairs in \underline{w} such that $\mathcal{G}_A(\underline{w}') = \mathcal{T}$. Since, in general, this can be done in a nonunique way, the integrals on the RHS contain the same \underline{w} more than once, and the inequality is strict unless f is concentrated on minimally spanning \underline{w} .

We apply this inequality with $f(\underline{w}) := \prod_{w \in \underline{w}} \lambda^2 C h(v-u)$ to (10.38) (with the integrand dominated as indicated there). The resulting integral over $d\underline{w}''$ can then be performed similarly to (10.36), yielding $e^{C|A|}$. This proves the claim. \square

We now derive the main result of the present section

Proposition 10.1. *For λ small enough and with $\epsilon_0 = C|\lambda|^\alpha$,*

$$\sum_{A \subset \mathbb{N} : \min A = 1} \epsilon_0^{-|A|} \text{dist}(A)^\alpha \|G_A^c\|_{10\gamma_0} \leq 1 \quad (10.40)$$

Proof. Each \underline{w} that spans A minimally determines a spanning tree on A . Hence we can reorganize the bound (10.37) by first integrating all \underline{w} that determine the same tree. This amounts to integrate, for each edge of the tree, all pairs (u, v) that determine this edge. Furthermore, we use that

$$\text{dist}(A)^\alpha \leq \prod_{\{\tau, \tau'\} \in \mathcal{E}(\mathcal{T})} (1 + |\tau' - \tau|)^\alpha, \quad \text{for any spanning tree } \mathcal{T} \text{ on } A \quad (10.41)$$

where $\mathcal{E}(\mathcal{T})$ is the set of edges of the spanning tree \mathcal{T} (see Appendix B for a simple proof). We arrive at the bound

$$\text{dist}(A)^\alpha \|G_A^c\|_{10\gamma_0} \leq e^{C|A|} \sum_{\text{span. trees } \mathcal{T} \text{ on } A} \prod_{(\tau, \tau') \in \mathcal{E}(\mathcal{T})} \hat{e}_\alpha(\tau, \tau') \quad (10.42)$$

where, for $\tau < \tau'$,

$$\hat{e}_\alpha(\tau, \tau') := \lambda^2 (1 + |\tau - \tau'|)^\alpha \int_{\text{Dom}(\tau)} du \int_{\text{Dom}(\tau')} dv h(v-u), \quad (10.43)$$

and $\hat{e}_\alpha(\tau', \tau) := \hat{e}_\alpha(\tau, \tau')$. Next, we establish

$$\sum_{\tau' \in \mathbb{N} \setminus \{\tau\}} \hat{e}_\alpha(\tau, \tau') \leq C|\lambda|^{2\alpha}, \quad (10.44)$$

by using the bound (10.30) on h from Lemma 10.1;

$$\begin{aligned}\hat{e}_\alpha(\tau, \tau+1) &\leq \lambda^2 C \int_0^{2\lambda^{-2}t_0} ds sh(s) \leq \lambda^2 C (2\lambda^{-2}t_0)^{1-\alpha} \leq C|\lambda|^{2\alpha} \\ \sum_{\tau' > \tau+1} \hat{e}_\alpha(\tau, \tau') &\leq \lambda^2 C (t_0|\lambda|^{-2})^{-\alpha} \int_{\lambda^{-2}(\tau-1)t_0}^{\lambda^{-2}\tau t_0} du \int_{u+\lambda^{-2}t_0}^{\infty} dv (v-u)^\alpha h(v-u) \leq C|\lambda|^{2\alpha}\end{aligned}$$

Starting from the inequality (10.42), we bound the sum in (10.40) as

$$\sum_{\substack{A \subset \mathbb{N} \\ \min A = 1}} \epsilon_0^{-|A|} \text{dist}(A)^\alpha \|G_A^c\|_{10\gamma_0} \leq \sum_{\substack{\text{trees } \mathcal{T} \text{ on } \mathbb{N} : \\ \text{Supp}(\mathcal{T}) \ni 1, |\mathcal{T}| \geq 2}} (C|\lambda|^{2\alpha})^{-|\mathcal{E}(\mathcal{T})|} \prod_{(\tau, \tau') \in \mathcal{E}(\mathcal{T})} \hat{e}_\alpha(\tau, \tau') \quad (10.45)$$

where we denote by $\text{Supp}(\mathcal{T})$ the vertices that have at least one edge and $\mathcal{E}(\mathcal{T})$ is the set of edges. This sum can be controlled relying on the bound (10.44). This is a special case of Lemma B.1 in Appendix B where (10.44) plays the role of the Kotecky-Preiss criterion (B3), $\kappa = 1$, and the edges $\{\tau, \tau'\}$ of the tree \mathcal{T} play the role of the sets S . \square

10.3 Bounds on boundary correlation functions

In this section, we work in the equilibrium case $\beta_1 = \beta_2 = \beta$, since only in that case the boundary correlation functions are relevant. We recall from Section 4.2.2:

$$\check{Z}_t \rho_S = \text{Tr}_E [e^{-itH} D_{\text{rd}}(\rho_S \otimes \rho_E^{\text{ref}}) D_{\text{rd}}^* e^{itH}] \quad (10.46)$$

which differs from the reduced evolution Z_t by the fact that we included the Radon-Nikodym derivative D_{rd} .

For a dense set of $\rho \in \mathcal{B}_1(\mathcal{H})$, we can write a formal Duhamel expansion

$$D_{\text{rd}} \rho D_{\text{rd}}^* = e^{\Delta F(\beta)} \sum_{m=0}^{\infty} \int_{0 < \beta_1 < \dots < \beta_m < \beta} d\beta_1 \dots d\beta_m \lambda^m \tilde{L}_I(\beta_m) \dots \tilde{L}_I(\beta_1) \rho \quad (10.47)$$

where the term with $m = 0$ is defined to be 1 and

$$\tilde{L}_I(\beta_i) \rho = -\frac{1}{2} [e^{\frac{\beta_i}{2}(H_S + H_E)} H_I e^{-\frac{\beta_i}{2}(H_S + H_E)}, \rho]_+ \quad (10.48)$$

with $[B, A]_+ = BA + AB$. We will now combine this expansion with the Dyson expansion for the unitary evolution to obtain an expansion for \check{Z}_t . For this it is convenient to set in eq. (10.47) $\beta_i = t_i - \beta$ with $t_i \in [-\beta, 0]$ and

$$L_I(t) := \tilde{L}_I(t + \beta) \quad \text{for } t \in [-\beta, 0].$$

We also generalize

$$U_J := U_{J \cap [0, t]} \quad \text{for } J \subset [-\beta, t].$$

and

$$K_{u,v} := (i1_{[u \geq 0]} + 1_{[u < 0]})(i1_{[v \geq 0]} + 1_{[v < 0]}) \mathbb{E}[L_I(v) \otimes_S L_I(u)] \quad (10.49)$$

This bizarre looking formula simply takes care of the fact that the Dyson expansion for the Radon Nikodym derivative does not have factors of i . In particular, the above definitions of $U_J, K_{u,v}$ reduces to the ones given previously in Section 10.1.1 in the case $J \in \mathbb{R}_+, 0 < u < v$. For a set of pairs $\underline{u} \in \Omega_{[-\beta, t]}$, we let now $\mathcal{J}(\underline{u})$ be the induced collection of intervals partitioning the interval $[-\beta, t]$ instead of the interval $[0, t]$.

It is now straightforward to check the formal expansion

$$\check{Z}_t = e^{\Delta F(\beta)} \int_{\Omega_{[-\beta, t]}} d\underline{w} \mathcal{T} \left[\bigotimes_{w \in \underline{w}} K_w \bigotimes_{J \in \mathcal{J}_{[-\beta, t]}(\underline{w})} U_J \right] \quad (10.50)$$

The factor $e^{\Delta F(\beta)}$ can be determined from the relation

$$1 = \text{Tr}[\check{Z}_0 \rho_S^{\text{ref}}] = \text{Tr}[\check{Z}_0 (1_{[x=0]} \nu^{\text{ref}})] \quad (10.51)$$

where ρ_S^{ref} and ν^{ref} were defined in Section 4.2.1 (note that $1_{[x=0]} \nu^{\text{ref}}$ is a function on \mathbb{A}_0) and the last equality exploits the fact that ρ_S^{ref} and ν^{ref} do not depend on $x \in \mathbb{X}_0$ and also the operator \check{Z}_0 is translation-invariant. The advantage of the rightmost term in (10.51) is that \check{Z}_0 acts on an operator in $\mathcal{B}_1(\mathcal{H}_S)$ that is strictly localized on the lattice and in particular it has a limit as $\Lambda \nearrow \infty$. Let us first define

$$\check{h}(s) = \sup_{-\beta \leq u \leq v: v-u=s} \|K_{u,v}\|_{20\gamma_0} \quad (10.52)$$

Then, Assumption A still implies⁷ that $\int_{\mathbb{R}_+} ds (1 + s^\alpha) \check{h}(s) < C$. To see this, let us inspect how the operator $K_{u,v}$ written out explicitly in (10.25), gets modified for negative u, v .

- 1) Depending on $a, b \in \{0, 1\}$, there are minus signs that do not affect our bounds.
- 2) The summand in (2.16) gets multiplied with $e^{\pm r \omega_j(q)}$, $0 < r < \beta$. Since the function ω_j is smooth and bounded, this does not affect the decay properties of $K_{u,v}$ for large $v - u$.
- 3) The operators $\mathbb{1}_x^a$ in (10.25) should be replaced by $e^{r H_S} \mathbb{1}_x^a e^{-r H_S}$, $0 < r < \beta/2$. By a simple propagation estimate in imaginary time, cfr. (10.33), the $\|\cdot\|_{20\gamma_0}$ -norm of such expressions is uniformly bounded by a constant.

By the above remarks, it now follows that Lemma 10.1 still holds if we replace $h \rightarrow \check{h}$. In particular, we can bound the expansion in (10.50) as in Lemma 10.1, we control the expansion of the rightmost term in (10.51) and hence we obtain the thermodynamic limit of the number $e^{\Delta F(\beta)}$ and the equilibrium state ν^β , and we prove that the operator $T_1(0)$ defined in Section 4.2.2 is bounded and has a thermodynamic limit, too. Finally, from the expansion we get as well that

$$\sup_{s \in \mathcal{S}} \left(e^{20\gamma_0|v|} |\mu^\beta(s) - \mu^{\text{ref}}(s)| \right) \leq C\lambda^2 \quad (10.53)$$

with $\mu^\beta(s), \mu^{\text{ref}}(s)$ defined in Section 4.2.1 and we recall that $s = (v, \eta, e_L, e_R)$.

Let us next generalize the expression for the correlation functions G_A^c . For the macroscopic time $\tau = 0$, we define $\text{Dom}(\tau) := [-\beta, 0]$. Then the equalities (10.21) and (10.23) remain true without any changes and Lemma 10.2 remains true upon replacing $h \rightarrow \check{h}$.

Finally, we prove the bound (6.18) (Induction hypothesis 6.2) at scale $n = 0$.

Proposition 10.2. *Recall that $\epsilon_0 = C|\lambda|^\alpha$ and $\epsilon_{1,0} = C|\lambda|^{2-2\alpha}$. For λ small enough;*

$$\sum_{\substack{A \subset \mathbb{N} \\ \min A = 0}} \epsilon_0^{-|A|} \text{dist}(A)^\alpha \|G_A^c\|_{20\gamma_0} \leq \epsilon_{1,0} \quad (10.54)$$

Proof. The proof mimics that of Proposition 10.1. We first derive

$$\text{dist}(A)^\alpha \|G_A^c\| \leq e^{C|A|} \sum_{\text{span. trees } \mathcal{T} \text{ on } A} \prod_{(\tau, \tau') \in \mathcal{E}(\mathcal{T})} \hat{e}_\alpha(\tau, \tau') \quad (10.55)$$

⁷Of course, this is true provided that we treat β as a fixed constant and we do not attempt to get bounds that are uniform as $\beta \rightarrow \infty$.

where $\hat{e}_\alpha(\tau, \tau')$ was defined in (10.43) and this definition carries over to the case where one of τ, τ' equals 0, provided that we replace again h by \check{h} . However, the bound (10.44) can now be improved as

$$\sum_{\tau' \in \mathbb{N} \setminus \{\tau\}} \hat{e}_\alpha(\tau, \tau') \leq \begin{cases} C|\lambda|^{2\alpha} & \tau \neq 0 \\ C\lambda^2 & \tau = 0 \end{cases} \quad (10.56)$$

the improvement for $\tau = 0$ is due to the fact that the length of $\text{Dom}(\tau)$ is $\beta < C$ for $\tau = 0$ instead of $t_0\lambda^{-2}$ for $\tau \neq 0$. In the sum over trees (10.45), there is now always one edge of the form $\{0, \tau\}$ for which we have the bound $C\lambda^2 = C\epsilon_0^2\epsilon_{1,0}$ instead of $C\epsilon_0^2$, and this accounts trivially for the improved bound on the RHS of (10.54). \square

11 Estimates on the first scale: reduced evolution T

The analysis of the operator $T = T_{n=0}$ proceeds in three steps. Note that T depends on the coupling constant λ both through the strength of the interaction and through the definition of the time scale $\lambda^{-2}t_0$, since $T = Z_{\lambda^{-2}t_0}$. In a first step, accomplished in Section 11.1, we prove that, as $\lambda \rightarrow 0$, the operator $Z_{\lambda^{-2}t}$ can be replaced by the Markov semigroup $e^{t(-i\lambda^{-2}L_S + M)}$ with M a dissipative operator. This is the 'Markov approximation' that we referred to already in Section 2.4.3. This Markov approximation is well-known in the literature since the work of Davies [7] and we describe it mainly for reasons of completeness.

In the second step, in Section 11.2, we argue that the Markov semigroup has good properties, corresponding more or less to the requirements that Prop 6.3 imposes on T . The reasoning in that section is standard in the analysis of (classical) linear Boltzmann equations, involving standard tools as Weyl's theorem and the Perron-Frobenius theorem. Finally, in Section 11.3, we argue that T inherits these properties from the Markov semigroup. This follows immediately by perturbation theory.

11.1 The weak-coupling limit

11.1.1 The generator M

We first define the generator M and we investigate its properties. Let U_t^0 stand for the propagator U_t with $\lambda = 0$, that is;

$$U_t^0 = e^{-itL_{\text{spin}}}, \quad L_{\text{spin}} = \text{ad}(H_{\text{spin}}) \quad (11.1)$$

and we will again write U_J^0 to denote $U_{s'-s}^0$ for an interval $J = [s, s']$. We define

$$\lambda^2 G_{v-u}^0 := \mathcal{T} [K_{u,v} \otimes U_{[u,v]}^0] \quad (11.2)$$

$$\lambda^2 G_{v-u} := \mathcal{T} [K_{u,v} \otimes U_{[u,v]}] \quad (11.3)$$

Since U_t^0 is a finite dimensional unitary tensored with the identity, acting only on the spin degrees of freedom, $\|U_t^0\|_\gamma \leq C$ uniformly in t, γ , and (10.30) implies

$$\int_0^\infty ds (1 + |s|)^\alpha \|G_s^0\|_{20\gamma_0} \leq C. \quad (11.4)$$

Next, let P_ε be the projector to the eigenspace corresponding to the eigenvalue $\varepsilon \in \mathcal{E} = \sigma(L_{\text{spin}})$, cfr. eq. (2.21). Then the connection between the operators G_s^0 and the generator M is via

$$M = \sum_{\varepsilon \in \mathcal{E}} \int_0^\infty ds e^{-is\varepsilon} P_\varepsilon G_s^0 P_\varepsilon \quad (11.5)$$

Thus we also have

$$\|M\|_{20\gamma_0} \leq C. \quad (11.6)$$

11.1.2 Emergence of M from the Dyson expansion

We recall the expansion (10.12) for the reduced evolution Z_t :

$$Z_t = \int_{\Omega_{[0,t]}} d\underline{w} \mathcal{T} \left[\bigotimes_{w \in \underline{w}} K_w \bigotimes_{J \in \mathcal{J}(\underline{w})} U_J \right] \quad (11.7)$$

We will show that the leading contribution to Z_t in the above integral comes from of pairs

$$\Omega_{[0,t]}^0 := \{\underline{w} \in \Omega_{[0,t]} : v_i < u_{i+1} \ i = 1, \dots, |\underline{w}| - 1\}. \quad (11.8)$$

Set $\Omega_{[0,t]}^1 = \Omega_{[0,t]} \setminus \Omega_{[0,t]}^0$ and fix $\gamma = 20\gamma_0$ in Lemmata 11.1 and 11.2. First we have

Lemma 11.1.

$$\int_{\Omega_{[0,t]}^1} d\underline{w} \left\| \mathcal{T} \left[\bigotimes_{w \in \underline{w}} K_w \bigotimes_{J \in \mathcal{J}(\underline{w})} U_J \right] \right\|_{\gamma} \leq C e^{C\lambda^2 t} |\lambda|^{2\alpha} \quad (11.9)$$

Proof. We proceed as in the proof of Lemma 10.2, and we bound the LHS of (11.9) by

$$e^{\lambda^2 C t} \int_{\Omega_{[0,t]}^1} d\underline{w} \prod_{w \in \underline{w}} \lambda^2 h(v - u) \quad (11.10)$$

Every $\underline{w} \in \Omega_{[0,t]}^1$ has to contain at least two pairs $(u, v), (u', v')$ such that $u < u'$ but $u' < v$. Choose the first two such pairs and integrate over the coordinates of all other pairs, using that $\|h\|_1 = \int ds h(s) < \infty$ and proceeding as in the proof of Lemma 10.1. This yields the bound

$$(11.10) \leq e^{\lambda^2 t(C + \|h\|_1)} q(t) \quad (11.11)$$

with

$$q(t) = \int_{0 < u < u' < v < t, u' < v' < t} h(v - u) h(v' - u')$$

and $q(t)$ can be easily estimated by first performing the integral over v' , which gives a factor $\|h\|_1$, and then the one over u' , such that one gets (recall that $\alpha \leq 1$)

$$q(t) \leq \|h\|_1 \lambda^4 \int_{0 < u < v < t} du dv h(v - u) |v - u| \quad (11.12)$$

$$\leq \|h\|_1 \lambda^4 \int_0^t du \left(\max_{0 < s < t-u} |s|^{1-\alpha} \right) \int ds |s|^\alpha h(s) \quad (11.13)$$

$$\leq \|h\|_1 \left(\int ds |s|^\alpha h(s) \right) (\lambda^2 t)^2 |\lambda|^{2\alpha} \quad (11.14)$$

which yields the bound (11.9) upon invoking Assumption A. \square

Next, we organize the contributions from leading sets of pairs. We call Z_t^0 the integral over them, i.e. in eq. (11.7) we replace $\Omega_{[0,t]}$ by $\Omega_{[0,t]}^0$. Note that these can be written explicitly as

$$Z_t^0 = \sum_{n=0}^{\infty} \lambda^{2n} \int d\underline{u} d\underline{v} \dots U_{u_3-v_2} G_{v_2-u_2} U_{u_2-v_1} G_{v_1-u_1} U_{u_1} \quad (11.15)$$

where G_s and U_s were defined above. We prove

Lemma 11.2.

$$\|Z_t^0 - e^{(-iL_S + \lambda^2 M)t}\|_\gamma \leq C e^{\lambda^2 C t} |\lambda|^{2\alpha} \quad (11.16)$$

Proof. Using integrability of G_t and the bound (10.33) together with the product structure in eq. (11.15), we get that the Laplace transform

$$\hat{Z}_z^0 := \int_0^\infty dt e^{-tz} Z_t^0 \quad (11.17)$$

is analytic in the half plane $\operatorname{Re} z > C\lambda^2$ and given explicitly by

$$\hat{Z}_z^0 = \left(z + iL_{\text{spin}} + i\lambda^2 m_p^{-1} \operatorname{ad}(\Delta) - \lambda^2 \hat{G}_z \right)^{-1} \quad (11.18)$$

where $\hat{G}_z = \int_0^\infty dt e^{-tz} G_t$ and we recalled that $L_S = L_{\text{spin}} + \lambda^2 m_p^{-1} \operatorname{ad}(\Delta)$. Using (10.30) we get \hat{G}_z is analytic in $\operatorname{Re} z > C|\lambda|^2$ too and bounded there by

$$\|\hat{G}_z\|_\gamma \leq C. \quad (11.19)$$

The same bound holds for $\operatorname{ad}(\Delta)$. Recall that the spectrum of L_{spin} is real and given by the set \mathcal{E} (2.21). Hence

$$\|\hat{Z}_z^0\|_\gamma \leq \frac{1}{\operatorname{dist}(z, i\mathcal{E})} \quad (11.20)$$

if $\operatorname{Re} z > C|\lambda|^2$. By the resolvent identity

$$\hat{Z}_z^0 - (z + iL_{\text{spin}})^{-1} = \lambda^2 \hat{Z}_z^0 (i m_p^{-1} \operatorname{ad}(\Delta) - \hat{G}_z) (z + iL_{\text{spin}})^{-1} \quad (11.21)$$

which implies, using (11.20) and $\operatorname{Re} z > C|\lambda|^2$,

$$\|\hat{Z}_z^0 P_\varepsilon\|_\gamma \leq C \frac{1}{|z + i\varepsilon|} \quad (11.22)$$

where we recall that P_ε is the spectral projector to eigenvalue ε eigenspace of L_{spin} .

Inverse Laplace transform gives

$$Z_t^0 - e^{(-iL_S + \lambda^2 M)t} = \int_{\mathcal{C}} dz e^{tz} (\hat{Z}_z^0 - (z + iL_S - \lambda^2 M)^{-1}) \quad (11.23)$$

where $\mathcal{C} = \{z : \operatorname{Re} z = A|\lambda|^2\}$ and A is taken large enough, $A > C$. By the resolvent identity we have

$$\hat{Z}_z^0 - (z + iL_S - \lambda^2 M)^{-1} = \lambda^2 \hat{Z}_z^0 (\hat{G}_z - M) (z + iL_S - \lambda^2 M)^{-1} \quad (11.24)$$

and from (11.5)

$$M = \sum_{\varepsilon} e^{i\varepsilon} P_\varepsilon \hat{G}_0^0 P_\varepsilon.$$

The integrand in (11.23) is large when z is close to $-i\varepsilon$, $\varepsilon \in \mathcal{E}$. Thus we localize the contour $\mathcal{C} = \cup_{\varepsilon \in \mathcal{E}} \mathcal{C}_\varepsilon$ such that on \mathcal{C}_ε , $-i\varepsilon$ is the nearest element of $-i\mathcal{E}$. For $z \in \mathcal{C}_\varepsilon$ we write

$$\hat{G}_z - M = (\hat{G}_z - \hat{G}_z^0) + (\hat{G}_z^0 - \hat{G}_{i\varepsilon}^0) + (\hat{G}_{i\varepsilon}^0 - M) := D_1^\varepsilon + D_2^\varepsilon + D_3^\varepsilon. \quad (11.25)$$

and set

$$\lambda^2 \hat{Z}_z^0 D_i^\varepsilon (z + iL_S - \lambda^2 M)^{-1} := I_i^\varepsilon(z) \quad (11.26)$$

Then, by (11.21),

$$Z_t^0 - e^{(-iL_S + \lambda^2 M)t} = \sum_{\varepsilon} \int_{\mathcal{C}_\varepsilon} dz e^{tz} (I_1^\varepsilon(z) + I_2^\varepsilon(z) + I_3^\varepsilon(z)). \quad (11.27)$$

Recalling (11.2) and (11.3) and using $\|U_t - U_t^0\|_\gamma \leq e^{C|\lambda|^2 t} - 1$ together with $\operatorname{Re} z = A|\lambda|^2$, A large enough, we bound

$$\|D_1^\varepsilon\|_\gamma \leq \int_0^\infty dt h(t) |e^{-C|\lambda|^2 t} - 1| \leq \sup_t \left(\frac{|e^{-C|\lambda|^2 t} - 1|}{(1+t)^\alpha} \right) \int_0^\infty dt h(t) (1+t)^\alpha \leq C|\lambda|^{2\alpha}. \quad (11.28)$$

where we used (10.30).

To bound $\|D_2^\varepsilon\|_\gamma$ we note that by (11.4) \hat{G}_z^0 is analytic in $\operatorname{Re} z > 0$ and by the same bound as in (11.28), we get for $z \in \mathcal{C}_\varepsilon$

$$\|D_2^\varepsilon\|_\gamma \leq C \int_0^\infty dt h(t) |e^{-zt} - e^{i\varepsilon}| \leq C \min\{1, |z + i\varepsilon|^{2\alpha}\} \quad (11.29)$$

Since (11.20) also holds for the resolvent of $-iL_S + \lambda^2 M$

$$\left\| \int_{\mathcal{C}_\varepsilon} dz e^{tz} (I_1^\varepsilon(z) + I_2^\varepsilon(z)) \right\|_\gamma \leq C|\lambda|^2 \int_{\mathcal{C}_\varepsilon} dz e^{zt} \frac{1}{|z + i\varepsilon|^2} (|\lambda|^{2\alpha} + \min\{1, |z + i\varepsilon|^{2\alpha}\}) \leq C e^{C\lambda^2 t} |\lambda|^{2\alpha}. \quad (11.30)$$

Finally, to estimate I_3^ε , from (11.5) we have $M = \sum_{\varepsilon'} P_{\varepsilon'} \hat{G}_{i\varepsilon}^0 P_{\varepsilon'}$ and thus

$$D_3^\varepsilon = \hat{G}_{i\varepsilon}^0 - M = \sum_{(\varepsilon_1, \varepsilon_2) \neq (\varepsilon, \varepsilon)} P_{\varepsilon_1} \hat{G}_{i\varepsilon}^0 P_{\varepsilon_2} - \sum_{\varepsilon' \neq \varepsilon} P_{\varepsilon'} \hat{G}_{i\varepsilon}^0 P_{\varepsilon'}$$

Using (11.22) and the analogous estimate for $P_\varepsilon(z + iL_S - \lambda^2 M)^{-1}$ we get

$$\|I_3^\varepsilon(z)\|_\gamma \leq C\lambda^2 \sum_{(\varepsilon_1, \varepsilon_2) \neq (\varepsilon, \varepsilon)} (|z + i\varepsilon_1| |z + i\varepsilon_2|)^{-1}$$

and so

$$\left\| \int_{\mathcal{C}_\varepsilon} e^{tz} I_3(z) dz \right\|_\gamma \leq C|\lambda|^2 \sum_{(\varepsilon_1, \varepsilon_2) \neq (\varepsilon, \varepsilon)} \int_{\mathcal{C}_\varepsilon} dz e^{zt} (|z + i\varepsilon_1| |z + i\varepsilon_2|)^{-1} \leq C e^{C\lambda^2 t} |\lambda|^2 \log |\lambda| \quad (11.31)$$

The bounds (11.30) and (11.31) yield the claim. \square

For completeness, we summarize Lemmata 11.1 and 11.2 in a single statement that is widely known as Davies' weak coupling limit.

Proposition 11.1. *For any $t < \infty$, and λ small enough,*

$$\sup_{t < \lambda^{-2} t} \|Z_t - e^{(-iL_S + \lambda^2 M)t}\|_{20\gamma_0} \leq C e^{Ct} |\lambda|^{2\alpha} \quad (11.32)$$

11.2 Analysis of the semigroup e^{tQ}

We will now analyze the Markov semigroup $e^{t(-iL_S + \lambda^2 M)}$ that was derived in the previous section. The splitting of the generator into L_S and $\lambda^2 M$ is logical from the point of view of the derivation from the microscopic dynamics, since M represents the contribution due to the interaction with the environment, but it is not optimal for the analysis of the semigroup, instead, we write

$$-L_S + \lambda^2 M = -L_{\text{spin}} + \lambda^2 Q, \quad \text{with } Q := -\operatorname{ad}\left(\frac{1}{m_p} \Delta\right) + M \quad (11.33)$$

such that on the RHS, the first term is of $\mathcal{O}(1)$ and the second is proportional to λ^2 . Moreover, both terms commute and consequently it suffices to investigate the semigroup $t \mapsto e^{t\lambda^2 Q}$, or simply $t \mapsto e^{tQ}$.

11.2.1 Fourier transform revisited

Note that up to this point, we have viewed ρ as a function $\rho(x, s)$ on \mathbb{A}_0 and M and Q as kernels $K(x', s'; x, s)$ on $\mathbb{A}_0 \times \mathbb{A}_0$. In particular, translation invariant kernels were studied in terms of the Fourier transform $\hat{K}(p; s, s')$ in the variable $x' - x$. For the present discussion, it is easier to use a slightly different, in fact more natural, Fourier representation. Namely we replace the previously used position variable $\lfloor \frac{x_L + x_R}{2} \rfloor$ by $\frac{x_L + x_R}{2}$. Recall from Section 2.4.1 that ρ can be also be viewed as a kernel $\rho(x_L, x_R) \in \mathcal{B}(\mathcal{S})$, with $x_L, x_R \in \mathbb{Z}^d$. Similarly, any K in \mathcal{R} can be viewed as a kernel

$$K(x'_L, x'_R, x_L, x_R) \in \mathcal{B}(\mathcal{B}(\mathcal{S})). \quad (11.34)$$

Define for $\rho \in \mathcal{B}_2(\mathcal{H}_S)$

$$\tilde{\rho}(p, k) = \sum_{x_L, x_R} e^{ip \frac{x_L + x_R}{2}} e^{ik(x_L - x_R)} \rho(x_L, x_R) \quad (11.35)$$

which is in $L^2(\mathbb{T}^d \times \mathbb{T}^d, \mathcal{B}_2(\mathcal{S}))$. Denote by $\tilde{\rho}(p) := \tilde{\rho}(p, \cdot)$.

In Section 5.3.2, we defined the Fourier transform $\hat{\rho}(p) := \hat{\rho}(p, \cdot)$ wrt. to the variable $x = \lfloor \frac{x_L + x_R}{2} \rfloor$. The two transforms are closely related, namely a short calculation gives

$$\tilde{\rho}(p) = \mathcal{I}_p \hat{\rho}(p)$$

where $\mathcal{I}_p : l^2(\mathcal{S}) \rightarrow L^2(\mathbb{T}^d, \mathcal{B}_2(\mathcal{S}))$ is the unitary map

$$(\mathcal{I}_p f)(k) = \sum_{v, \eta} f(v, \eta) e^{ip \frac{\eta}{2}} e^{ik(2v + \eta)}. \quad (11.36)$$

where on the RHS, $f(v, \eta) \in \mathcal{B}_2(\mathcal{S})$, and we recall that v, η are the (position-like) coordinates that, together with $e_L, e_R \in \sigma(H_{\text{spin}})$, constitute the variable $s \in \mathcal{S}$.

For a translation invariant kernel K with $\|K\|_\gamma < \infty$ for some $\gamma > 0$, we have as before

$$(\tilde{K}\rho)(p) = \tilde{K}(p)\tilde{\rho}(p). \quad (11.37)$$

where $\tilde{K}(p) \in \mathcal{B}(L^2(\mathbb{T}^d, \mathcal{B}_2(\mathcal{S})))$ (cfr. the Remark in Section 5.2.2). Since Q is translation-invariant and $[Q, L_{\text{spin}}] = 0$, as we see from (11.5), we can decompose further

$$\tilde{Q}(p) = \oplus_{\varepsilon \in \mathcal{E}} \tilde{Q}_\varepsilon(p) \quad (11.38)$$

where $\tilde{Q}_\varepsilon(p) := P_\varepsilon \tilde{Q}(p) P_\varepsilon$ with P_ε the spectral projections of L_{spin} . By Assumption C, the spaces $P_\varepsilon(\mathcal{B}(\mathcal{S})), \varepsilon \neq 0$ are one-dimensional and hence we can identify

$$\tilde{Q}_\varepsilon(p) \in \mathcal{B}(L^2(\mathbb{T}^d)), \quad \varepsilon \neq 0$$

The space $P_0(\mathcal{B}(\mathcal{S}))$ has $\dim \mathcal{S}$ -dimensions and there is a natural basis labelled by $e \in \sigma(H_{\text{spin}})$, such that $P_0(L^2(\mathbb{T}^d, \mathcal{B}_2(\mathcal{S})))$ is identified with $L^2(\mathcal{F})$, with $\mathcal{F} = \sigma(H_{\text{spin}}) \times \mathbb{T}^d$ and so

$$\tilde{Q}_0(p) \in \mathcal{B}(L^2(\mathcal{F})).$$

11.2.2 Lindblad representation for Q

We can write a more explicit expression for M of (11.5) using (11.2) and (10.25):

$$(M\rho)(x, y) = \sum_{\varepsilon} \left(\zeta_\varepsilon(x - y) W_\varepsilon^* \rho(x, y) W_\varepsilon - \int_0^\infty ds \zeta(0, s) e^{-is\varepsilon} W_\varepsilon W_\varepsilon^* \rho(x, y) - \int_{-\infty}^0 ds \zeta(0, s) e^{-is\varepsilon} \rho(x, y) W_\varepsilon W_\varepsilon^* \right). \quad (11.39)$$

In (11.39) we have denoted $W_\varepsilon := P_\varepsilon(W)$, ζ_ε is the Fourier transform of ζ defined in (2.22) and we use the notation $\rho(x, y) \in \mathcal{B}(\mathcal{S})$ with $x, y \in \mathbb{Z}^d$ (see Section 2.2). To derive (11.39) from (10.25) and (11.2), one needs to calculate

$$P_\varepsilon \mathcal{T}((W \otimes \mathbb{1}_x)^a \otimes U_t^0 \otimes (W \otimes \mathbb{1}_y)^b) P_\varepsilon = P_\varepsilon W^a U_t^0 W^b P_\varepsilon \otimes \mathbb{1}_x^a \mathbb{1}_y^b.$$

To proceed write P_ε in terms of the projectors $\pi_e \in \mathcal{B}(\mathcal{S})$ to the basis vector labelled by $e \in \sigma(H_{\text{spin}})$. For $0 \neq \varepsilon = e_1 - e_2$ we have $P_\varepsilon \rho = \pi_{e_1} \rho \pi_{e_2}$ and for $\varepsilon = 0$ we have $P_0 \rho = \sum_e \pi_e \rho \pi_e$. Some algebra then gives the representation (11.39).

Denote the operator acting on ρ in the first term of (11.39) by Φ :

$$(\Phi \rho)(x, y) := \sum_\varepsilon \zeta_\varepsilon(x - y) W_\varepsilon^* \rho(x, y) W_\varepsilon. \quad (11.40)$$

One can easily check that Φ is a completely positive map.

In the second term of (11.39), using $\overline{\zeta(0, s)} = \zeta(0, -s)$ we have

$$\int_0^\infty ds e^{-is\varepsilon} \zeta(0, s) = (1/2)\zeta_\varepsilon(0) + it_\varepsilon(0)$$

with $t_\varepsilon(0)$ real, and in the third term one has $\int_{-\infty}^0 ds e^{-is\varepsilon} \zeta(0, s) = (1/2)\zeta_\varepsilon(0) - it_\varepsilon(0)$. Thus altogether we may write (11.39) as

$$M\rho = \Phi(\rho) - \frac{1}{2}(\Phi^*(\mathbb{1})\rho + \rho\Phi^*(\mathbb{1})) + i[H_{\text{Lamb}}, \rho] \quad (11.41)$$

where Φ^* is the adjoint of Φ w.r.t. to the trace, $\text{Tr } \rho\Phi^*(O) = \text{Tr } O\Phi(\rho)$ (whereas $*$ in W_ε^* denotes the Hermitian adjoint in $\mathcal{B}(\mathcal{S})$)) i.e.

$$\Phi^*(\mathbb{1}) = \sum_\varepsilon \zeta_\varepsilon(0) W_\varepsilon W_\varepsilon^* \quad (11.42)$$

and the "Lamb shift" to the Hamiltonian by

$$H_{\text{Lamb}} := \sum_\varepsilon t_\varepsilon(0) W_\varepsilon W_\varepsilon^*. \quad (11.43)$$

In terms of the Fourier transform variables of Section 11.2.1 we have

$$(\widetilde{\Phi \rho})(p, k) = \sum_\varepsilon \int \hat{\zeta}_\varepsilon(dq) W_\varepsilon \tilde{\rho}(p, k - q) W_\varepsilon^* \quad (11.44)$$

where the positive measure $\hat{\zeta}_\varepsilon(dq)$ is the Fourier transform of the function $\zeta_\varepsilon(x)$, introduced in (2.24). Note that $\tilde{\Phi}(p)$ is independent of p and hence we will write

$$\tilde{\Phi} = \tilde{\Phi}(p).$$

Let us next decompose as in (11.38). We get

$$\tilde{\Phi}_\varepsilon = 0 \quad \text{if } \varepsilon \neq 0.$$

Indeed, writing $\varepsilon = e_1 - e_2$, we have $P_\varepsilon(W_{\varepsilon'}^*(P_\varepsilon \rho)W_{\varepsilon'}) = \pi_{e_1} W_{\varepsilon'}^* \pi_{e_1} \rho \pi_{e_2} W_{\varepsilon'} \pi_{e_2}$ which obviously unless $\varepsilon' = 0$. However, for $\varepsilon' = 0$, $\zeta_{\varepsilon'} = 0$ by Assumption C.

Consider then $\tilde{\Phi}_0$. As explained in the previous section it can be viewed as acting on functions $\varphi(e, k)$ on $\sigma(H_{\text{spin}}) \times \mathbb{T}^d$. Recalling the definition (2.25) of the jump rates j , eq. (11.44) becomes

$$(\tilde{\Phi}_0 \varphi)(e', k') = \sum_{e \in \sigma(H_{\text{spin}})} \int j(e', dk; e, 0) \varphi(e, k' - k) \quad (11.45)$$

and since $\hat{\zeta}_\varepsilon$ is a finite positive measure, $\tilde{\Phi}_0$ defines a bounded positivity preserving operator on $L^1(\mathcal{F})$, $L^\infty(\mathcal{F})$, and, by Riesz-Thorin interpolation, also on $L^q(\mathcal{F})$, $1 < q < \infty$.

Denote the escape rates of the Markov process by

$$w(e) := \sum_{e'} \int j(e', dk'; e, 0) \quad (11.46)$$

Then we end up with the following explicit decomposition (11.38):

Proposition 11.2. (a) For $\varepsilon = 0$

$$\tilde{Q}_0(p) = \tilde{\Phi}_0 + q_0(p) \quad (11.47)$$

where $\tilde{\Phi}_0$ is a compact operator on $L^q(\mathcal{F})$, $q \geq 1$. It is also positivity improving, i.e. if $\varphi \geq 0$, then $e^{t\tilde{\Phi}_0}\varphi > 0$ for any $t > 0$. $q_0(p)$ is a multiplication operator on $L^2(\mathcal{F})$ by the function

$$q_0(p)(e, k) = -w(e) + iE_{\text{kin}}(p, k) \quad (11.48)$$

where $E_{\text{kin}}(p, k) = \frac{2}{m_p} \sum_{j=1}^d (\cos(\frac{p_j}{2} + k_j) - \cos(\frac{p_j}{2} - k_j))$ and

$$\min w := \min_e w(e) > 0. \quad (11.49)$$

(b) For $\varepsilon = e - e' \neq 0$, $Q_\varepsilon(p)$ is a multiplication operator on $L^q(\mathcal{F})$ by the function

$$q_\varepsilon(p)(e, k) = -1/2(w(e) + w(e')) + iE_{\text{kin}}(p, k) + iq(\varepsilon) \quad (11.50)$$

with $q(\varepsilon)$ real.

Proof. Compactness of $\tilde{\Phi}_0$ follows from the fact the function $x \rightarrow \zeta_\varepsilon(x)$ decays at infinity for any $\varepsilon \in \mathcal{E}$ by Assumption C. $e^{t\tilde{\Phi}_0}$ is positivity improving since by the same assumption the Markov process is irreducible. For the same reason the escape rates are strictly positive, i.e. (11.49).

In (11.48) the first term on the RHS comes from the $\Phi^*(\mathbb{1})$ terms in (11.41) and the second one from the Laplacian term in (11.33). In (11.50) the last term comes from the last term in (11.41). This term is zero for $\varepsilon = 0$. \square

11.2.3 Spectral properties of $\tilde{Q}_\varepsilon(p)$

We can now state the main result on spectral properties of $\tilde{Q}_\varepsilon(p)$:

Proposition 11.3. There are strictly positive constants $\gamma_Q, \mathfrak{p}_Q, a_Q, b_Q$ s.t. the following holds.

(a) $\tilde{Q}_\varepsilon(p)$ extend to an analytic family of bounded operators on $L^1(\mathcal{F})$ for $\varepsilon = 0$ and on $L^1(\mathbb{T}^d)$ for $\varepsilon \neq 0$ for $|\text{Im } p| < \gamma_Q$.

(b) Take p with $|\text{Re } p| \leq \mathfrak{p}_Q$ and $|\text{Im } p| < \gamma_Q$. Then the spectrum of $\tilde{Q}_0(p)$ lies in the half plane $\text{Re } z < -a_Q$ except for a simple isolated eigenvalue at $f_Q(p)$ with $\text{Re } f_Q(p) > -a_Q/2$. Moreover $f_Q(p)$ is analytic and satisfies

$$|f_Q(p) + D_Q p^2| \leq C|p|^4 \quad (11.51)$$

for a strictly positive 'diffusion constant' $D_Q > 0$. The corresponding left eigenvector can be chosen to be analytic in p such that, at $p = 0$, it equals $1_{\mathcal{F}}$ and the right one $\mu_Q(k, e)$ such that, at $p = 0$, $\mu_Q(k, e)$ is constant in k , positive and bounded from below. The spectrum of $\tilde{Q}_\varepsilon(p)$ lies in the half plane $\text{Re } z < -a_Q$.

(c) Take p with $|\text{Re } p| \geq \mathfrak{p}_Q$ and $|\text{Im } p| < \gamma_Q$. Then $\sigma(\tilde{Q}_\varepsilon(p)) \subset \{z : \text{Re } z \leq -b_Q\}$ for all ε .

(d) The claims (a)- (c) remain true for the operators $e^{\kappa \partial_k} \tilde{Q}_\varepsilon(p) e^{-\kappa \partial_k}$ for $\kappa \in \mathbb{C}$, $|\kappa| \leq \kappa_0$ for some $\kappa_0 > 0$.

Proof. The analyticity follows since by Proposition 11.2 the only p -dependence of $\tilde{Q}_\varepsilon(p)$ is in the multiplication operators which are analytic. The spectral claim for $\varepsilon \neq 0$ follows from Proposition 11.2 (b) since the real part of the multiplication operator is positive.

Let now $\varepsilon = 0$. Since $\tilde{\Phi}_0$ is compact and the spectrum of q_0 lies in a half space $\{z : \text{Re } z \leq -b\}$ with $b > 0$ the spectrum of $\tilde{Q}_0(p)$ in $\{z : \text{Re } z \geq -b/2\}$ consists of a finite number of eigenvalues of finite multiplicity. Since $\tilde{Q}_0(0)$ generates a Markov semigroup the spectrum of $\tilde{Q}_0(0)$ lies in $\{z : \text{Re } z \leq 0\}$. Since the Markov process is irreducible, 0 is a simple eigenvalue and all other eigenvalues have negative real parts. By analyticity for γ_Q, \mathfrak{p}_Q small enough the simple eigenvalue persists and lies in $\text{Re } z > -a_Q/2$ whereas the rest of spectrum is in $\text{Re } z < -a_Q$ for some $a_Q > 0$. Since the semigroup is positivity improving by the Perron-Frobenius theorem the corresponding

eigenfunction $\mu_Q(k, e)$ can be chosen to be strictly positive. It is the density of the unique invariant state of the Markov process on \mathcal{F} . From the fact that the jump rate depends on k, k' only through $k' - k$, it follows that $\mu_Q(k + g, e)$ is also invariant i.e. that μ_Q is constant in k .

Let's turn to the claim (11.51). By perturbation theory of isolated eigenvalues, we get (recall that $\langle \cdot, \cdot \rangle$ is the pairing between L^1 and L^∞ .)

$$f_Q(p) = \langle 1, (Q_0(p) - Q_0(0))\mu_Q \rangle + O(|p|^2) \quad (11.52)$$

The first term on the RHS vanishes by the explicit expression (11.48) and the fact that μ_Q is constant in k . To determine the second order contribution to $f_Q(p)$ we invoke second order spectral perturbation theory, yielding

$$f_Q(p) = - \sum_{i,j=1}^d p_i p_j \langle 1, v_j Q_0(0)^{-1} v_i \mu_Q \rangle + O(|p|^4), \quad (11.53)$$

where $v_i = -i\partial_{p_i} E_{\text{kin}}(p, \cdot)$ is the "group velocity" (2.39). The inverse of $Q_0(0)$ in this formula is well-defined because $R_Q(0)v_i\mu_Q = 0$ where

$$R_Q(p) = |\mu_Q(p)\rangle\langle 1| \quad (11.54)$$

is the spectral projection to μ_Q .

The non-negativity of D_Q is deduced from the fact that the operators \tilde{Q}_p (for $\text{Im } p = 0$) generate contractive semigroups. To establish the strict positivity of D_Q , we employ a standard construction: First, consider the space $L^2(\mathcal{F}, \mu_Q)$, defined by the scalar product $\langle \psi, \psi' \rangle_{\mu_Q} := \langle \mu_Q \psi, \psi' \rangle$ where $\psi\mu_Q$ is the pointwise product of two functions, and let the operator K be defined by

$$\langle \psi, K\psi' \rangle_{\mu_Q} = \langle \mu_Q Q_0(0)^* \psi, \psi' \rangle \quad (11.55)$$

where $Q_0(0)^*$ is the adjoint of $Q_0(0)$ acting on L^∞ . By using that the density μ_Q is bounded below and the explicit expression for the jump rates $j(\cdot; \cdot)$, we check that the operator K is bounded and sectorial, that is, there is a constant $C_K > 0$ such that

$$|\langle \psi, (\text{Im } K)\psi \rangle_{\mu_Q}| \leq -C_K \langle \psi, (\text{Re } K)\psi \rangle_{\mu_Q} \quad (11.56)$$

where $2\text{Re } K = K + K^*$, $2\text{Im } K = K - K^*$ and the adjoint is taken in the Hilbert space $L^2(\mathcal{F}, \mu_Q)$. Both the left and right eigenvectors of K are 1 (constant function on \mathcal{F}) and the diffusion constant can be represented as

$$D_Q \delta_{i,j} = \langle v_i, K^{-1} v_j \rangle_{\mu_Q} \quad (11.57)$$

Moreover, K inherits the unicity of the eigenvalue at 0 and the spectral gap from $\tilde{Q}_0(0)$ and the sectoriality allows to conclude that $D_Q \neq 0$. In the case where $\beta_1 = \beta_2$, the operator K is self-adjoint and the above reasoning can be slightly simplified. Hence the claims for $\tilde{Q}_0(0)$ hold.

Consider then $\tilde{Q}_0(p)$ for $p \neq 0$. From (11.48) we see that $\tilde{Q}_0(p)$ is the sum of two bounded generators of contractive semigroups on $L^1(\mathcal{F})$. Hence it generates a contractive semigroup as well, as follows by the Trotter product formula. We need to exclude point spectrum on the imaginary axis. Thus, assume that $\tilde{Q}_0(p)$ has an eigenvalue on the imaginary axis, λ_p . Since the spectrum of $\tilde{Q}_0(p)$ is discrete in the neighborhood of the imaginary axis, λ_p must be an eigenvalue of the adjoint $\tilde{Q}_0^*(p)$, acting on $L^\infty(\mathcal{F})$. The adjoint thus has at least one eigenvector $\varphi \in L^\infty(\mathcal{F})$. Such an eigenvector can be extended to an eigenvector of the adjoint of Q , Q^* , acting on $\mathcal{B}(\mathcal{H}_S)$. Indeed, let

$$S_{\varphi,p} = O_\varphi e^{ipX} \in \mathcal{B}(\mathcal{H}_S) \quad (11.58)$$

where $O_\varphi \in \mathcal{B}(\mathcal{H}_S)$ is the operator that acts by multiplication with $\phi(e, k)$ on functions in $\mathcal{H}_S \sim L^2(\mathcal{F})$ and X is the operator acting by multiplication with $x \in \mathbb{Z}^d$ on $l^2(\mathbb{Z}^d)$. By definition of the Fourier representation

$$\text{Tr}[e^{ipX} O_\varphi \rho] = \sum_{e \in \sigma(H_{\text{spin}})} \int dk \phi(e, k) (P_0 \tilde{\rho})(p, k, e) = \langle \phi, P_0 \tilde{\rho}(p) \rangle \quad (11.59)$$

where we wrote $\langle \cdot, \cdot \rangle$ for the pairing between functions in $L^\infty(\mathcal{F})$ and $L^1(\mathcal{F})$. Hence, for all $\rho \in \mathcal{B}_1(\mathcal{H}_S)$

$$\text{Tr}[S_{\varphi,p} Q \rho] = \langle \varphi, \tilde{Q}_0 P_0 \tilde{\rho}(p) \rangle = \langle \tilde{Q}_0^*(p) \varphi, P_0 \tilde{\rho}(p) \rangle = \lambda_p \langle \varphi, P_0 \tilde{\rho}(p) \rangle = \lambda_p \text{Tr}[S_{\varphi,p} \rho] \quad (11.60)$$

and it follows that $Q^*S_{\varphi,p} = \lambda_p S_{\varphi,p}$. In Section 11.2.4 we show that this equation has a non-trivial solution only when $p = 0$.

Finally, for (c) note that the operator ∂_k commutes with $\tilde{\Phi}_0$ so that

$$\tilde{Q}_\varepsilon^\kappa(p) := e^{\kappa\partial_k} \tilde{Q}_\varepsilon(p) e^{-\kappa\partial_k} = \tilde{Q}_\varepsilon(p) + i(E_{\text{kin}}(p, k - \kappa) - E_{\text{kin}}(p, k)) \quad (11.61)$$

Hence by perturbation theory (a) and (b) remain valid if $|\kappa|$ is small enough. \square

11.2.4 Perron-Frobenius theorem for maps on $\mathcal{B}(\mathcal{H}_S)$

Definition 11.1. A positive map ϕ on $\mathcal{B}(\mathcal{H}_S)$ is ergodic if for any $A \geq 0$, $A \neq 0$, $e^{t\phi} A > 0$ for any $t > 0$.

Proposition 11.4 (Schrader [23]). *Assume that a positive map ϕ on $\mathcal{B}(\mathcal{H}_S)$ is ergodic and that its spectral radius $r(\phi)$ is an eigenvalue with corresponding eigenvector S . Then $r(\phi)$ is a simple eigenvalue and S can be chosen positive definite: $S > 0$.*

We will apply the above theorem to the map e^{tQ^*} , with $t > 0$ arbitrary. First we establish

Lemma 11.3. *The map Φ^* is ergodic on $\mathcal{B}(\mathcal{H}_S)$.*

Proof. By duality, it suffices to prove that $e^{t\Phi}(\rho) > 0$ for any nonnegative $\rho \in \mathcal{B}_1$. Employing the fiber decomposition, we see that this is equivalent to $e^{t\tilde{\Phi}_{\varepsilon=0}}\varphi > 0$ for any nonnegative function φ on \mathcal{F} . This was proven in Proposition 11.2. \square

Lemma 11.4. *The map e^{tQ^*} is ergodic on $\mathcal{B}(\mathcal{H}_S)$.*

Proof. First, we note that the generator Q^* has the form

$$Q^*(O) = \Phi^*(O) + BO + OB^*, \quad B = i\left(\frac{1}{m_p}\Delta + H_{\text{Iamb}}\right) - \frac{1}{2}\Phi^*(\mathbb{1}) \quad (11.62)$$

Since $V_t : O \rightarrow e^{tB} O e^{tB^*}$ is a completely positive map, for any operator B , all integrands in the Duhamel expansion

$$e^{tQ^*} = 1 + \sum_{m=1}^{\infty} \int_{0 < t_1 < \dots < t_m < t} dt_m \dots dt_1 Q_t(t_1, \dots, t_m) \quad (11.63)$$

with $Q_t(t_1, \dots, t_m) := V_{t-t_m} \Phi^* \dots \Phi^* V_{t_2-t_1} \Phi^* V_{t_1}$ are completely positive maps, as well.

Therefore, for any $O \in \mathcal{B}(\mathcal{H}_S)$, it suffices to find an m_0 and a subset of positive measure of the m_0 -dimensional simplex such that $Q_t(t_1, \dots, t_m)(S) > 0$. Let m_0 be smallest natural number m such that $(\Phi^*)^m O > 0$. The ergodicity of Φ^* ensures that $m_0 < \infty$. Then by continuity, there is a neighborhood of $t_1 = t_2 = \dots = t_{m_0} = 0$ in the simplex where the integrand is bounded away from 0. \square

11.3 Proof of Proposition 6.1 for $n = 0$

Recall that $T_0 = Z_{\lambda^{-2}t_0}$. From Proposition 11.1, the relation $L_S + \lambda^2 M = L_{\text{spin}} + \lambda^2 Q$ and the fact that L_{spin} and Q , we infer

$$\|T_0 - e^{-i\lambda^{-2}t_0 L_{\text{spin}}} e^{t_0 Q}\|_{20\gamma_0} \leq C|\lambda|^{2\alpha}. \quad (11.64)$$

Then we set

$$T_Q := e^{-i\lambda^{-2}t_0 L_{\text{spin}}} e^{t_0 Q} = \sum_{\varepsilon} e^{-it_0 \lambda^{-2} \varepsilon} P_{\varepsilon} e^{t_0 Q} P_{\varepsilon}. \quad (11.65)$$

We need to translate the spectral information of Proposition 11.3 to the γ -norm. Recall that $e^{t_0 \hat{Q}(p)} = \mathcal{I}_p^{-1} e^{t_0 \tilde{Q}(p)} \mathcal{I}_p$ and that from (11.36) we have

$$e^{\kappa\partial_k} \mathcal{I}_p = \mathcal{I}_p e^{i\kappa(2v+\eta)}.$$

Thus

$$e^{t\hat{Q}(p)}(v', \eta'; v, \eta) = e^{i\kappa(2(v-v')+\eta-\eta')}(\mathcal{I}_p^{-1}e^{t\tilde{Q}^\kappa(p)}\mathcal{I}_p)(v', \eta'; v, \eta) \quad (11.66)$$

where we use the notation (11.61) and both sides are operators on $\mathcal{B}_2(\mathcal{S})$. Since

$$(\mathcal{I}_p^{-1}e^{t\tilde{Q}^\kappa(p)}\mathcal{I}_p)(v', \eta'; v, \eta) = \langle \varphi', e^{t\tilde{Q}^\kappa(p)}\varphi \rangle \quad (11.67)$$

where $\varphi(k) = e^{ip\frac{\gamma}{2}}e^{ik(2v+\eta)}$ (and φ' similarly) we conclude

$$\|P_\varepsilon e^{t\tilde{Q}(p)}P_\varepsilon\|_{\mathcal{G}} \leq C \sup_{|\kappa| \leq \gamma_0} \|e^{t\tilde{Q}_\varepsilon^\kappa(p)}\| \quad (11.68)$$

where the norm on the RHS is the operator norm on $\mathcal{B}(L^1(\mathbb{T}^d), \mathcal{B}(L^1(\mathcal{F}))$, depending on ε . By similar reasoning, we get a bound on the $\|\cdot\|_\gamma$ of the projection $R_Q(p)$.

We now finally choose $\gamma_0 := \min(\kappa_0, \gamma_Q)$ where the latter constants were introduced in Proposition 11.3. That proposition can then be recast as follows.

(1) Let p with $|\operatorname{Re} p| \leq \mathfrak{p}_Q$ and $|\operatorname{Im} p| < \gamma_0$. Then

$$T_Q(p) = e^{t_0 f_Q(p)} R_Q(p) + (1 - R_Q(p)) T_Q(p). \quad (11.69)$$

with

$$\|(1 - R_Q(p)) T_Q(p)\|_{\mathcal{G}} \leq C e^{-\frac{3a_Q}{4} t_0}, \quad (11.70)$$

and $f_Q(p)$ and $R_Q(p)$ are as in Proposition 11.3 with

$$|e^{t f_Q(p)}| \geq c e^{-\frac{a_Q}{2} t_0}. \quad (11.71)$$

(2) In the region $|\operatorname{Re} p| \geq \mathfrak{p}_Q$ and $|\operatorname{Im} p| < \gamma_0$

$$\|T_Q(p)\|_{\mathcal{G}} \leq C e^{-\frac{b_Q}{2} t_0}. \quad (11.72)$$

We can now apply spectral perturbation theory, e.g. Lemma A.1). First, for t_0 sufficiently large (compared to the constants in the above statements), the eigenvalue $e^{t_0 f_Q(p)}$ is isolated, by (11.70) and (11.71). Thus, taking then λ small enough, $|\lambda| \leq \lambda(t_0)$ the following holds.

(a) The isolated eigenvalue persists and the new eigenvalue $e^{f_0(p)}$ has the same properties: $f_0(0) = 0$ by unitarity, $\nabla f_0(0) = 0$ and $|f_0(p) + D_0 p^2| \leq C|p|^4$ by lattice symmetries, with $D_0 \rightarrow t_0 D_Q$ as $\lambda \rightarrow 0$.

(b) The bounds (11.70) and (11.72) hold for $e^{f_0(p)}$ and T_0 (and different C and c). Now take τ_0 large enough and then Proposition 6.1 (except for (6.12) and (6.4) for $\hat{T}_n(0, p)$) holds for $n = 0$ and $t_0 \in [\tau_0, 2\tau_0]$.

To get the claims for $R_0(0)$, i.e. (6.12), note that (10.53) tells us that μ^β is λ^2 -close to the $\lambda = 0$ system equilibrium state, i.e. the Gibbs state $\sim e^{-\beta H_{\text{spin}}}$. The latter is identical to the state $\mu_Q(0)$ in case $\beta_1 = \beta_2 = \beta$, and hence one has $\|P^\beta - R_Q(0)\|_{\mathcal{G}} = \mathcal{O}(\lambda^2)$. Since $\|R_Q(0) - R_0(0)\| = \mathcal{O}(|\lambda|^{2\alpha})$, (6.12) holds for $n = 0$. Finally, the bound (6.4) for $\hat{T}_0(0, p)$ was proven in Section 10.3.

A Appendix: Spectral perturbation theory

In the lemma below, we gather some spectral perturbation theory that is used in the proof of Lemma 7.3. We consider operators on a Banach space \mathcal{A} . Denote by $\|\cdot\|$ the usual operator norm on $\mathcal{B}(\mathcal{A})$ and let $\|\cdot\|_\diamond$ by a norm on (a dense subset of) $\mathcal{B}(\mathcal{A})$ such that

$$\begin{aligned} \|A\| &\leq \|A\|_\diamond \\ \|AB\|_\diamond &\leq \|A\|_\diamond \|B\|_\diamond \end{aligned}$$

The following lemma could just as well (and more naturally) be stated for $\|\cdot\|$ as for $\|\cdot\|_\diamond$. The use of $\|\cdot\|_\diamond$ is dictated by our application, where $\mathcal{A} = l^\infty(\mathcal{S})$ and $\|\cdot\|_\diamond = \|\cdot\|_\gamma$. In fact, in Lemma 7.3, we did not mention the space $\mathcal{A} = l^\infty(\mathcal{S})$ at all, but note that without such an underlying space, it is not a-priori clear what one means by expressions like ‘a simple eigenvalue’.

Lemma A.1. *Let the operator A_0 have a simple eigenvalue a_0 with corresponding one-dimensional spectral projection P_0 such that*

$$\|A_0 - a_0 P_0\|_\diamond < |a_0|, \quad \|P_0\|_\diamond < \infty \quad (\text{A1})$$

We study $A = A_0 + A_1$ for some perturbation A_1 . Define

$$b(r) := \frac{\|P_0\|_\diamond}{r} + \frac{1}{|a_0| - r - \|A_0 - a_0 P_0\|_\diamond}, \quad \text{for } r < |a_0| - \|A_0 - a_0 P_0\|_\diamond \quad (\text{A2})$$

If, for some $r > 0$, we have

$$\|A_1\|_\diamond b(r) < 1 \quad (\text{A3})$$

then the eigenvalue persists when the perturbation is added (we call it a) and

$$|a - a_0| \leq r \quad (\text{A4})$$

The spectral projection P corresponding to a satisfies

$$\|P - P_0\|_\diamond \leq 2\pi r b(r) \frac{\|A_1\|_\diamond b(r)}{1 - \|A_1\|_\diamond b(r)} \quad (\text{A5})$$

Proof. The condition (A1) implies that the eigenvalue a_0 is an isolated point in the spectrum. Let $\mathcal{C}_r \subset \mathbb{C}$ be the circle with radius r centered on a_0 and such that A_0 has no other spectrum within \mathcal{C}_r (the latter condition will be implied in our case by $\|A_0 - a_0 P_0\|_\diamond < r$). To prove that the eigenvalue persists within the circle \mathcal{C}_r , it suffices to show that $\sup_{z \in \mathcal{C}_r} \|(z - (A_0 + kA_1))^{-1}\|$ remains finite for $k \in [0, 1]$. Without loss, we can assume $k = 1$ in the proof below. Consider the Neumann series

$$(z - A)^{-1} = (z - A_0)^{-1} \sum_{n \geq 0} (A_1(z - A_0)^{-1})^n \quad (\text{A6})$$

To establish that the resolvent remains finite, we show a stronger property (because of the different norms), namely

$$\sup_{z \in \mathcal{C}_r} \sum_{n \geq 0} (\|A_1\|_\diamond \| (z - A_0)^{-1} \|_\diamond)^n < \infty \quad (\text{A7})$$

To get this, we bound

$$\sup_{z \in \mathcal{C}_r} \|(z - A_0)^{-1}\|_\diamond \leq \sup_{z \in \mathcal{C}_r} \frac{\|P_0\|_\diamond}{|z - a_0|} + \sup_{z \in \mathcal{C}_r} \|(z - (A_0 - a_0 P_0))^{-1}\|_\diamond \leq b(r) \quad (\text{A8})$$

where the last equality follows from the Neumann series. Therefore, (A7) is finite if $\|A_1\|_\diamond b(r) < 1$ and we conclude that the eigenvalue a lies within the circle \mathcal{C}_r . The bound for the projection follows as

$$P - P_0 = \int_{\mathcal{C}_r} dz (z - A)^{-1} - (z - A_0)^{-1} \quad (\text{A9})$$

$$= \int_{\mathcal{C}_r} dz (z - A_0)^{-1} \left(\sum_{n \geq 1} A_1 (z - A_0)^{-1} \right) \quad (\text{A10})$$

$$\|P - P_0\|_\diamond < \int_{\mathcal{C}_r} dz \|(z - A_0)^{-1}\|_\diamond \frac{\|A_1\|_\diamond \|(z - A_0)^{-1}\|_\diamond}{1 - \|A_1\|_\diamond \|(z - A_0)^{-1}\|_\diamond} \quad (\text{A11})$$

$$< 2\pi r b(r) \frac{\|A_1\|_\diamond b(r)}{1 - \|A_1\|_\diamond b(r)} \quad (\text{A12})$$

□

Let us denote by c, C constants that depend only on $\|P_0\|_\diamond$. If we assume that

$$\|A_1\|_\diamond \leq c(|a_0| - \|A_0 - a_0 P_0\|_\diamond), \quad (\text{A13})$$

then we can choose $r = C\|A_1\|_\diamond$ to estimate the spectral shift. To bound the difference of projections, we choose $r = c(|a_0| - \|A_0 - a_0 P_0\|_\diamond)$ and we obtain

$$\|P - P_0\|_\diamond \leq C \frac{\|A_1\|_\diamond}{|a_0| - \|A_0 - a_0 P_0\|_\diamond} \quad (\text{A14})$$

B Appendix: Combinatorics

We collect some easy combinatorial lemma's.

Lemma B.1. *Let \mathcal{T} be a spanning tree on the set $A \subset \mathbb{N}_0$, and define*

$$\text{dist}(\mathcal{T}) := \prod_{\{\tau, \tau'\} \in \mathcal{E}(\mathcal{T})} (1 + |\tau' - \tau|) \quad (\text{B1})$$

where $\mathcal{E}(\mathcal{T})$ is the set of edges of \mathcal{T} . Then

$$\text{dist}(A) \leq \text{dist}(\mathcal{T}) \quad (\text{B2})$$

with $\text{dist}(A)$ defined as in Section 6.

See e.g. [22] for the 5-line proof. The following result lies at the heart of cluster expansions. For subsets $S \in \mathbb{N}_0$, write $S \sim S'$ whenever $S \cap S' \neq \emptyset$. Let \mathcal{S} be a collection of subsets $\mathcal{S} = \{S_1, \dots, S_m\}$. Note that the connectedness of the graph with vertices S_j and edges $\{S_i, S_j\}$ whenever $S_j \sim S_i$ is equivalent to the connectedness of \mathcal{S} defined in Section 9 and, as in that section, we write \mathfrak{C} for the set of connected collections.

Proposition B.1. *Let $w(\cdot)$ be a function on subsets $S \in \mathbb{N}_0$. Assume that for some $\kappa > 0$,*

$$\sum_{S: S \sim S'} e^{\kappa|S|} |w(S)| \leq \kappa |S'|. \quad (\text{B3})$$

Then, with $w(\mathcal{S}) = \prod_{S \in \mathcal{S}} w(S)$,

$$\sum_{\mathcal{S}} 1_{[\{S'\} \cup \mathcal{S} \in \mathfrak{C}]} |w(\mathcal{S})| \leq e^{\kappa|S'|} \quad (\text{B4})$$

We refer to [25] for the proof. The condition (B3) is often called the Kotecky-Preiss condition. An immediate consequence is

$$\sum_{\mathcal{S} \in \mathfrak{C}: \text{Supp } \mathcal{S} \ni \tau} |w(\mathcal{S})| \leq \sum_{S': S' \ni \tau} |w(S')| \sum_{\mathcal{S}} 1_{[\{S'\} \cup \mathcal{S} \in \mathfrak{C}]} |w(\mathcal{S})| \leq \sum_{S': S' \ni \tau} |w(S')| e^{\kappa|S'|} \leq \kappa \quad (\text{B5})$$

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